ECET 4520

Industrial Distribution Systems, Illumination, and the NEC

Review

of

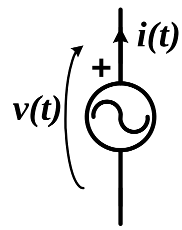
Complex Power

in

Steady-State AC Circuits

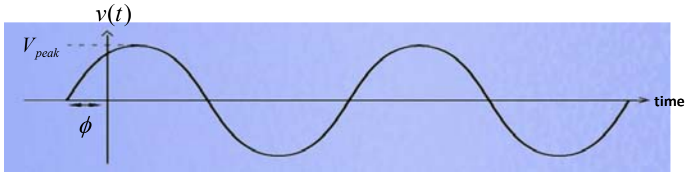
Steady-State AC Voltage Sources

The voltage potential of an AC source may be defined as:



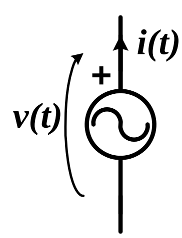
$$v(t) = V_{peak} \cdot \sin(\omega \cdot t + \phi)$$

where: V_{peak} is the peak value of the voltage,
ω is the angular frequency ($2\pi f$) of the waveform, and
ϕ is the phase angle of the voltage waveform.



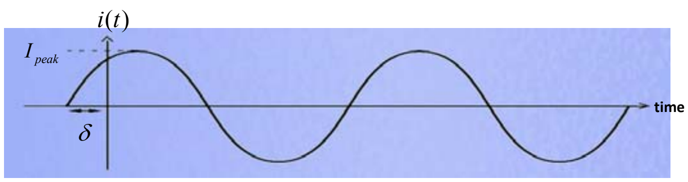
Steady-State AC Voltage Sources

Similarly, the current produced by the AC source may be defined as:



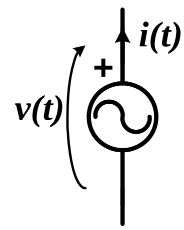
$$i(t) = I_{peak} \cdot \sin(\omega \cdot t + \delta)$$

where: I_{peak} is the peak value of the current,
ω is the angular frequency ($2\pi f$) of the waveform, and
δ is the phase angle of the current waveform.



Power in AC Circuits

In terms of electric circuits, **power** is defined as the **rate at which electric energy** is either produced or consumed within the circuit.

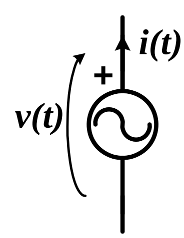


Although power is a "**rate**" of energy production or consumption and it is the energy that is either being produced or consumed, power itself is often referred to as either being produced or consumed.

Power in AC Circuits

Power may be calculated in terms of the voltage and current waveforms associated with a specific circuit element by:

$$p(t) = v(t) \cdot i(t) \quad \textit{(Watts)}$$



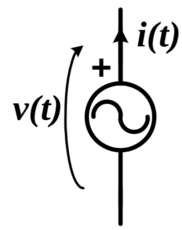
where: $p(t)$ is the instantaneous rate that an element either produces or consumes energy at any time t.

Power in AC Circuits

Note that the expression:

$$p(t) = v(t) \cdot i(t) \quad \textit{(Watts)}$$

defines the power "**produced**" by an element when the current is defined in the <u>same direction</u> as the voltage-rise across the element.

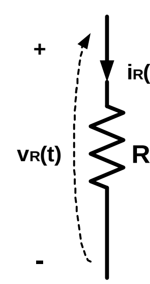


Power in AC Circuits

Note that the expression:

$$p(t) = v(t) \cdot i(t) \quad \textit{(Watts)}$$

defines the power "produced" by an element when the current is defined in the same direction as the voltage-rise across the element.



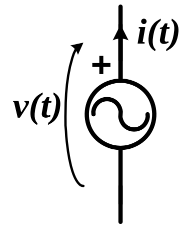
But, if the current is defined in the <u>opposite direction</u> as the voltage-rise across an element, then *p(t)* defines the power "**consumed**" by that element.

Power from an AC Source

In the case of an AC source where:

$$v(t) = V_{peak} \cdot \sin(\omega \cdot t + \phi)$$

$$i(t) = I_{peak} \cdot \sin(\omega \cdot t + \delta)$$



the general expression for power produced by the source is:

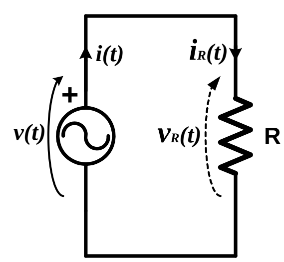
$$p(t) = v(t) \cdot i(t)$$
$$= V_{peak} \cdot I_{peak} \cdot \sin(\omega \cdot t + \phi) \cdot \sin(\omega \cdot t + \delta)$$

Power from an AC Source

The power expression:

$$p(t) = V_{peak} \cdot I_{peak} \cdot \sin(\omega \cdot t + \phi) \cdot \sin(\omega \cdot t + \delta)$$

is actually quite complex.



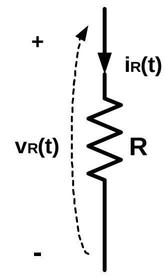
To better understand the true nature of the power expression, it may by useful to first consider the case where the voltage source is applied to a purely resistive load.

AC Sources and Resistive Loads

If the source is placed across a resistive load, the resultant current that will flow through the resistor is defined by Ohm's Law:

$$i_R(t) = \frac{v(t)}{R}$$

$$= \frac{V_{peak}}{R} \cdot \sin(\omega \cdot t + \phi)$$



AC Sources and Resistive Loads

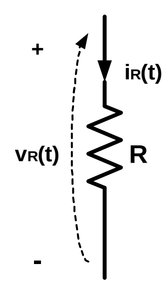
Thus, for a resistive load:

$$v_R(t) = V_{peak} \cdot \sin(\omega \cdot t + \phi)$$

$$i_R(t) = I_{peak} \cdot \sin(\omega \cdot t + \phi)$$

such that the peak value of the current is defined by the Ohm's Law relationship:

$$I_{peak} = \frac{V_{peak}}{R}$$



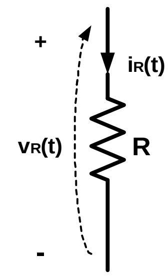
AC Sources and Resistive Loads

Thus, for a resistive load:

$$v_R(t) = V_{peak} \cdot \sin(\omega \cdot t + \phi)$$

$$i_R(t) = I_{peak} \cdot \sin(\omega \cdot t + \phi)$$

Note that the phase angle of the resistor's current is equal to the phase angle of the applied voltage…



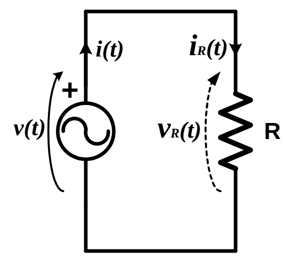
There is no phase shift between the voltage and current waveforms relating to a purely resistive load.

AC Power and Resistors

If the AC source is connected to a resistive load, then:

$$v_R(t) = V_{peak} \cdot \sin(\omega \cdot t + \phi)$$

$$i_R(t) = I_{peak} \cdot \sin(\omega \cdot t + \phi)$$



and the power consumed by the resistor is:

$$p_R(t) = v_R(t) \cdot i_R(t)$$

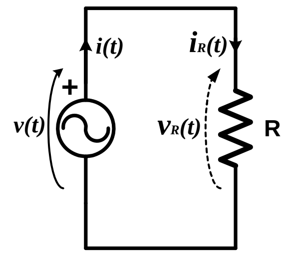
$$= V_{peak} \cdot I_{peak} \cdot \sin^2(\omega \cdot t + \phi)$$

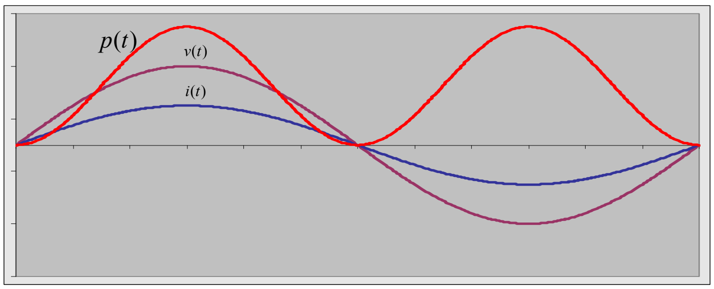
AC Power and Resistors

The solution for resistor power

$$p_R(t) = V_{peak} \cdot I_{peak} \cdot \sin^2(\omega \cdot t + \phi)$$

may be plotted as:

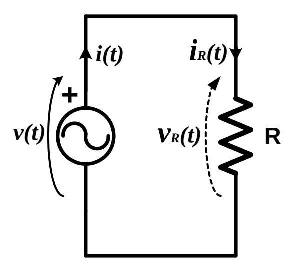


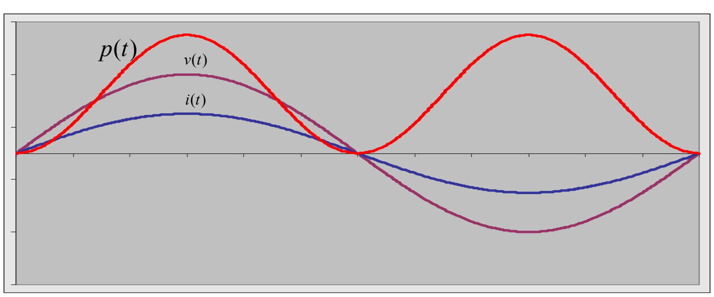


AC Power and Resistors

Note that power supplied to the resistor is always non-negative, which is expected since a resistor can only consume electric power.

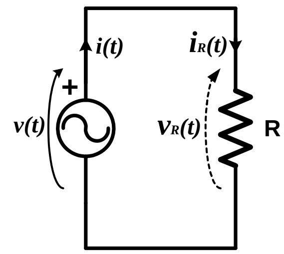
(Note – since *p(t)* is defined as power "consumed" by the resistor, a negative value would actually relate to power being "produced" by the resistor, which can not occur)

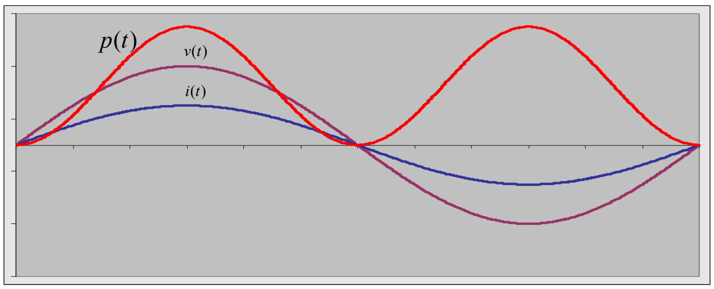




AC Power and Resistors

Looking at the plotted waveforms, it can be seen that the power supplied to the resistor also varies periodically, but at a frequency that is 2x larger than that of the applied voltage or the resultant current.



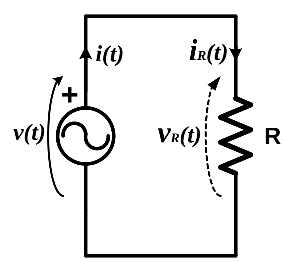


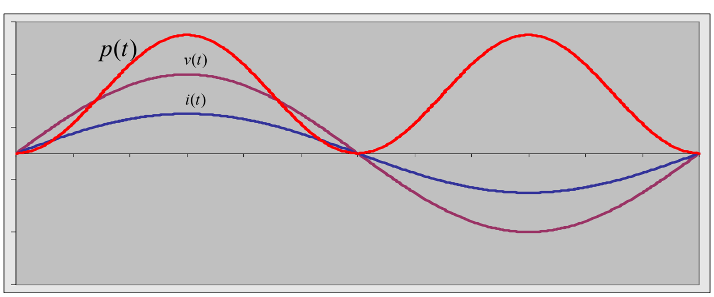
AC Power and Resistors

The peak value of the power waveform is:

$$P_{peak} = V_{peak} \cdot I_{peak}$$

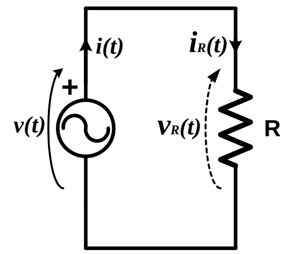
This should <u>not</u> be confused with the constant power provided to a resistor by a DC source.





AC Power and Resistors

To better understand the AC power waveform, it is useful to rewrite the power equation into the following form (by substituting the trigonometric identity $sin^2x = ½·[1 – cos2x]$):



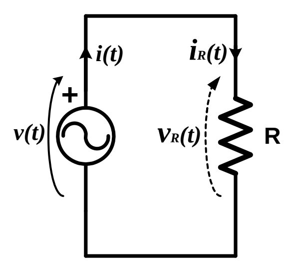
$$p_R(t) = V_{peak} \cdot I_{peak} \cdot \sin^2(\omega \cdot t + \phi)$$

$$= \frac{V_{peak} \cdot I_{peak}}{2} \cdot [1 - \cos(2 \cdot \omega \cdot t)]$$

$$= \frac{V_{peak} \cdot I_{peak}}{2} - \frac{V_{peak} \cdot I_{peak}}{2} \cdot \cos(2 \cdot \omega \cdot t)$$

AC Power and Resistors

Looking at the resultant AC power waveform:



$$p_R(t) = \boxed{\frac{V_{peak} \cdot I_{peak}}{2}} - \frac{V_{peak} \cdot I_{peak}}{2} \cdot \cos(2 \cdot \omega \cdot t)$$

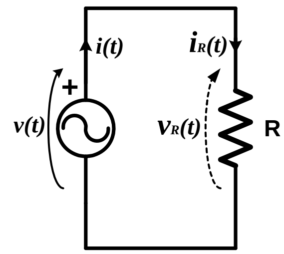
It can be seen that the waveform has two terms:

- The first term is a constant that relates to the **average** value of the power supplied to the resistor.

AC Power and Resistors

Looking at the resultant AC power waveform:

$$p_R(t) = \frac{V_{peak}\cdot I_{peak}}{2} - \boxed{\frac{V_{peak}\cdot I_{peak}}{2}\cdot\cos(2\cdot\omega\cdot t)}$$



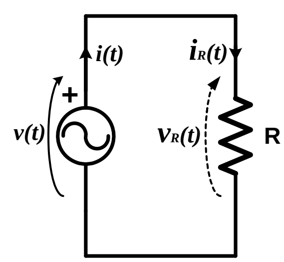
It can be seen that the waveform has two terms:

- The <u>second term</u> is a purely “sinusoidal” term that has a **zero average** value and varies at 2x the frequency of the source voltage.

Real Power

In AC systems, it is typically the <u>average value</u> of the power that is desired:

$$P_{R(AC)} = Avg[p_R(t)] = \frac{V_{peak}\cdot I_{peak}}{2} \quad \textit{(Watts)}$$

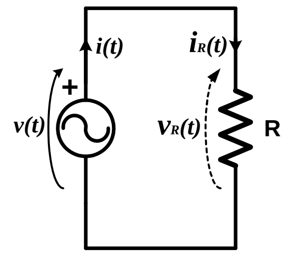


This average power value is called “**Real Power**”.

AC Power and Resistors

Note that the average AC power is only ½ that of the peak power value:

$$P_{R(AC)} = \frac{P_{peak}}{2} = \frac{V_{peak} \cdot I_{peak}}{2} \quad (Watts)$$

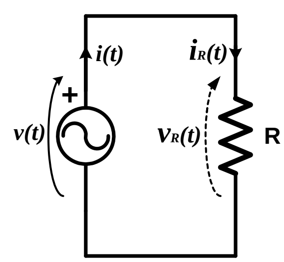


Although this result is expected since the source is time-varying, it provides a potentially confusing result if compared to power supplied by a DC source to a resistor:

$$P_{R(DC)} = V_{DC} \cdot I_{DC} \quad (Watts)$$

AC Power and Resistors

Comparing the results, it can be seen that an AC source, whose peak value is equal to the magnitude of a DC source ($V_{peak}=V_{DC}$), provides an average power to a resistor that is equal to only ½ of that provided by the DC source.



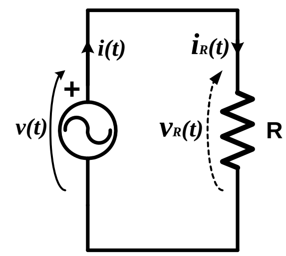
$$P_{R(AC)} = \frac{V_{peak} \cdot I_{peak}}{2} \quad (Watts)$$

$$P_{R(DC)} = V_{DC} \cdot I_{DC} \quad (Watts)$$

AC Power and Resistors

In other words:

In terms of power supplied to a resistor, an AC source is only ½ as <u>effective</u> as a DC source whose magnitude is equal to the peak value of the AC source.



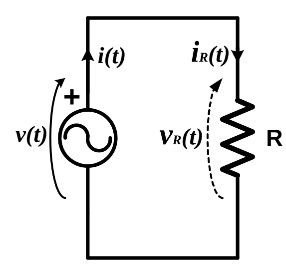
$$P_{R(AC)} = \frac{V_{peak} \cdot I_{peak}}{2} \quad (Watts)$$

$$If \;\; V_{peak} = V_{DC} \rightarrow$$

$$P_{R(DC)} = V_{DC} \cdot I_{DC} \quad (Watts)$$

Effective Voltage

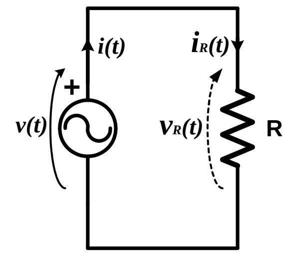
If an AC source is only ½ as <u>effective</u> as a DC source whose magnitude is equal to the peak value of the AC source, then an important question remains:



"Given an DC voltage source (V_{DC}) that supplies power to a resistor, if the DC source is replaced by an AC source, what peak voltage is required in order for the AC source to supply the same average power to the resistor as the DC source?"

Effective Voltage

Since the DC source supplies 2x more average power to a resistor than an AC source whose peak value is $V_{peak} = V_{DC}$, and since:



$$P_{R(AC)} \equiv V_{peak}^2 \ ,$$

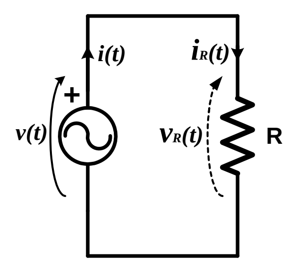
if the peak value of the AC source is increased by a factor of $\sqrt{2}$ to:

$$V_{peak} = \sqrt{2} \cdot V_{DC}$$

then the AC source will supply equal power to a resistor compared to the DC source.

Effective Voltage

Thus, if an **effective voltage** for an AC source is defined in comparison to the average power delivered to a resistor by a DC source, then an AC source having the peak value V_{peak} may be classified as having as an effective voltage whose magnitude is:



$$V_{effective} = \frac{V_{peak}}{\sqrt{2}}$$

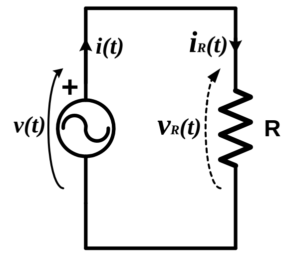
RMS Voltage Magnitude

The effective voltage:

$$V_{effective} = \frac{V_{peak}}{\sqrt{2}}$$

is equal to the **<u>RMS</u>** (***root-mean-squared***) value of the purely sinusoidal voltage, as defined by the function:

$$V_{RMS} = \sqrt{\frac{1}{T} \cdot \int_0^T v^2(t) \cdot dt} = \frac{V_{peak}}{\sqrt{2}}$$

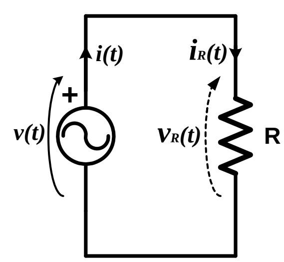


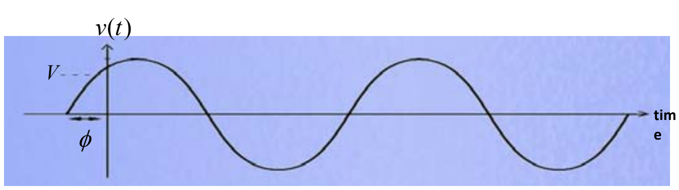
RMS Voltage Magnitude

The voltage waveform can be expressed in terms of its ***RMS voltage magnitude***:

$$v(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

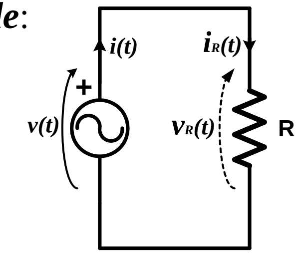
where: $V = \frac{V_{peak}}{\sqrt{2}}$ is the RMS magnitude of the AC voltage.





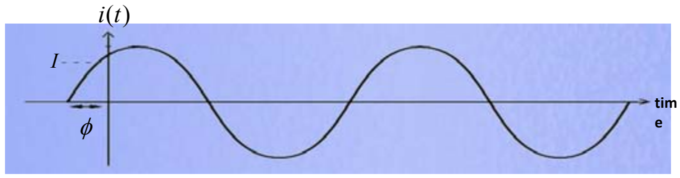
RMS Current Magnitude

Similarly, the current waveform can also be expressed in terms of its ***RMS current magnitude***:



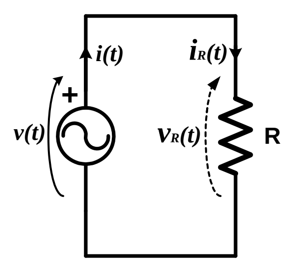
$$i(t) = \sqrt{2} \cdot I \cdot \sin(\omega \cdot t + \phi)$$

where: $I = \frac{I_{peak}}{\sqrt{2}}$ is the RMS magnitude of the AC current.



RMS Magnitudes & Resistor Power

When expressed in terms of their RMS magnitudes instead of their peak magnitudes:



$$V_{peak} = \sqrt{2} \cdot V \qquad I_{peak} = \sqrt{2} \cdot I$$

the power delivered to a resistor is:

$$p_R(t) = V \cdot I - V \cdot I \cdot \cos(2 \cdot \omega \cdot t)$$

with an average (Real Power) value of:

$$P_{R(AC)} = Avg[p_R(t)] = V \cdot I$$

RMS Magnitudes & Resistor Power

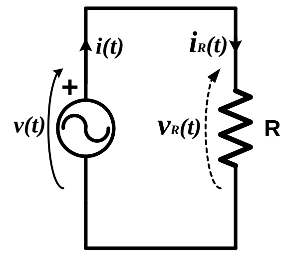
The result:

$$P_{R(AC)} = V \cdot I$$

is similar to the DC formula for power:

$$P_{DC} = V_{DC} \cdot I_{DC}$$

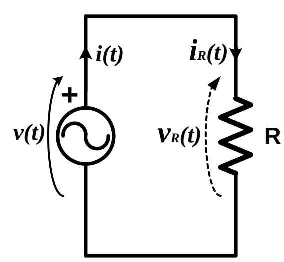
which provides an advantage for defining the AC waveforms in terms of their RMS magnitudes instead of their peak values.



Real Power

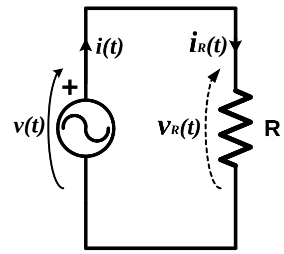
Real Power (*P*) is the average power produced or consumed by an element in an AC circuit.

Real Power is defined in units of ***Watts***.

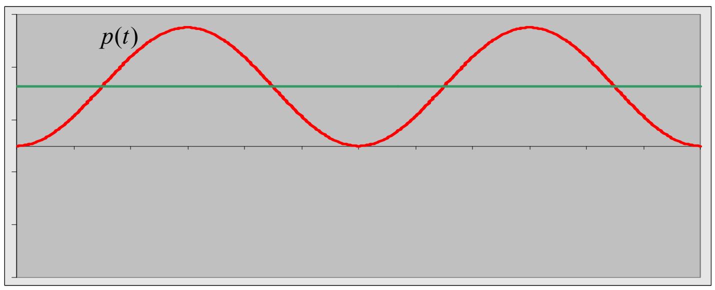


Real Power and Resistors

In a purely resistive AC circuit, if the voltages and currents are expressed in terms of their RMS magnitudes, then ***real power*** can be calculated as:



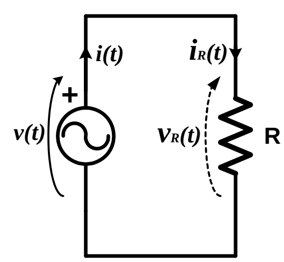
$$P = V \cdot I \quad Watts$$



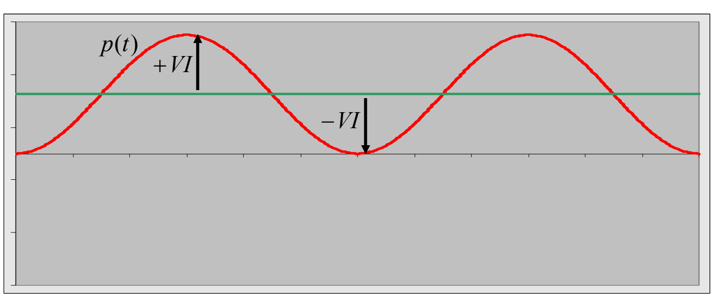
$P_{AVG} = V \cdot I$

Real Power and Resistors

Note that the instantaneous power waveform fluctuates above and below its average value by the amount V·I.



$$P = V \cdot I \quad Watts$$

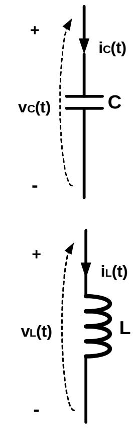


$P_{AVG} = V \cdot I$

AC Sources and Reactive Loads

What if the AC source is supplying a load that is purely reactive…

I.e. – either Capacitive or Inductive?



Similar to resistive loads, a sinusoidal (AC) voltage source will cause a sinusoidal (AC) current to flow through both capacitors and inductors.

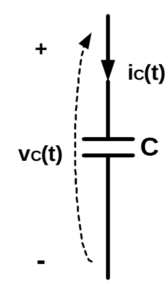
But, their voltage and current waveforms do not follow the linear Ohm's Law relationship. Instead, their voltage and current waveforms are governed by a differential relationship.

AC Sources and Capacitors

For an ideal capacitor, the voltage-current relationship is defined by the following equations:

$$i_C(t) = C \cdot \frac{dv_C(t)}{dt}$$

$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(t)dt = \frac{1}{C}\int_{0}^{t} i_C(t)dt + V_o$$



We may obtain a solution for steady-state AC operation from these relationships.

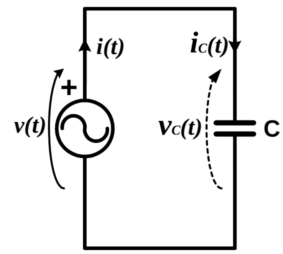
AC Sources and Capacitors

Given a sinusoidal voltage applied across a capacitor:

$$v_C(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

the associated capacitor current will be:

$$i_C(t) = \sqrt{2} \cdot V \cdot \omega \cdot C \cdot \cos(\omega \cdot t + \phi°)$$



To allow for direct comparison, the cosine function can be converted to an equivalent sine function using the identity:

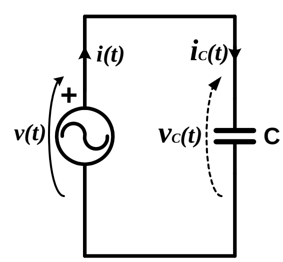
$$\cos(x) = \sin(x + 90°)$$

AC Sources and Capacitors

The resultant capacitor voltage and current waveforms, expressed as sine functions, are:

$$v_C(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

$$i_C(t) = \sqrt{2} \cdot V \cdot \omega \cdot C \cdot \sin(\omega \cdot t + \phi° + 90°)$$

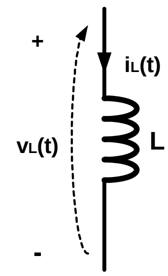


Note that:

- The capacitor current is phase-shifted by +90° compared to the capacitor voltage, and
- The voltage and current magnitudes do not follow the linear Ohm's Law relationship that holds true for resistors.

AC Sources and Inductors

For an ideal inductor, the voltage-current relationship is defined by the following equations:



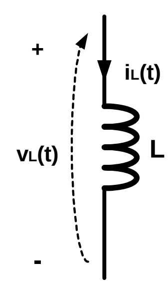
$$v_L(t) = L \cdot \frac{di_L(t)}{dt}$$

$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(t)dt = \frac{1}{L}\int_{0}^{t} v_L(t)dt + I_o$$

We may obtain a solution for steady-state AC operation from these relationships.

AC Sources and Inductors

Given the sinusoidal voltage applied to an inductor:



$$v_L(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

the associated current will be:

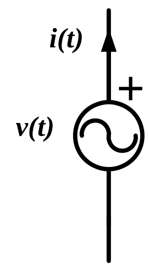
$$i_L(t) = \sqrt{2} \cdot \frac{V}{\omega \cdot L} \cdot \sin(\omega \cdot t + \phi° - 90°)$$

resulting in a power angle:

$$\theta = \phi - \delta = \phi - (\phi - 90) = +90°$$

AC Power – General Case

Now, lets go back and look at the general expression for power produced by the source:



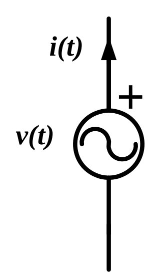
$$p(t) = v(t) \cdot i(t)$$
$$= V_{peak} \cdot I_{peak} \cdot \sin(\omega \cdot t + \phi) \cdot \sin(\omega \cdot t + \delta)$$

where: $v(t) = V_{peak} \cdot \sin(\omega \cdot t + \phi)$

$$i(t) = I_{peak} \cdot \sin(\omega \cdot t + \delta)$$

AC Power – General Case

When expressed in terms of the RMS magnitudes, the power expression becomes:



$$p(t) = v(t) \cdot i(t)$$
$$= \sqrt{2} \cdot V \cdot \sqrt{2} \cdot I \cdot \sin(\omega \cdot t + \phi) \cdot \sin(\omega \cdot t + \delta)$$
$$= 2 \cdot V \cdot I \cdot \sin(\omega \cdot t + \phi) \cdot \sin(\omega \cdot t + \delta)$$

since: $V_{peak} = \sqrt{2} \cdot V$ $\qquad I_{peak} = \sqrt{2} \cdot I$

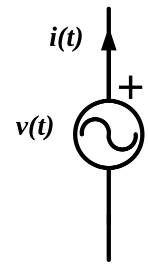
AC Power – General Case

The power expression:

$$p(t) = 2 \cdot V \cdot I \cdot \sin(\omega \cdot t + \phi) \cdot \sin(\omega \cdot t + \delta)$$

may be modified using several trigonometric identities into the following form:

$$\begin{aligned} p(t) = {} & V \cdot I \cdot \cos(\phi - \delta) \\ & - V \cdot I \cdot \cos(\phi - \delta) \cdot \cos(2 \cdot \omega \cdot t) \\ & + V \cdot I \cdot \sin(\phi - \delta) \cdot \sin(2 \cdot \omega \cdot t) \end{aligned}$$



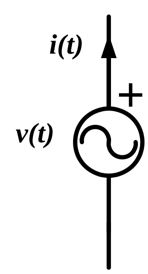
AC Power – General Case

The modified power expression is often simplified by defining a new variable, θ, where:

$$\theta = \phi - \delta$$

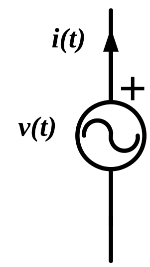
and substituting it into the equation:

$$\begin{aligned} p(t) = {} & V \cdot I \cdot \cos(\theta) \\ & - V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t) \\ & + V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t) \end{aligned}$$



AC Power – General Case

The angle θ, which is defined by the difference between the phase angles of the voltage and current, is often referred to as the "**power angle**":



$$\theta = \phi - \delta$$

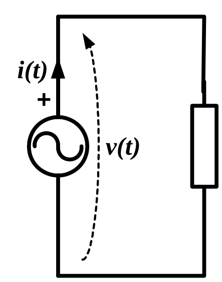
where: $$v(t) = V_{peak} \cdot \sin(\omega \cdot t + \phi)$$

$$i(t) = I_{peak} \cdot \sin(\omega \cdot t + \delta)$$

AC Source Power

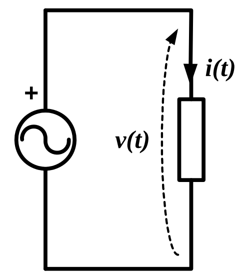
This solution defines the instantaneous power <u>produced by an AC source</u>.

$$p(t) = V \cdot I \cdot \cos(\theta) - V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t) + V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t)$$



AC Load Power

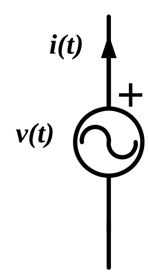
If the source is connected across a load (that may have resistive, capacitive, and/or inductive components), then the solution also provides the power consumed by the AC supplied load.



$$p(t) = V \cdot I \cdot \cos(\theta) - V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t) + V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t)$$

AC Power – General Case

The solution for AC power has three terms:



$$p(t) = V \cdot I \cdot \cos(\theta) - V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t) + V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t)$$

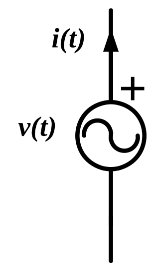
The first term is a **constant** that provides the average or "Real Power" value:

$$P = V \cdot I \cdot \cos(\theta)$$

AC Power – General Case

The solution for AC power has three terms:

$$p(t) = V \cdot I \cdot \cos(\theta)$$
$$- V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t)$$
$$+ V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t)$$



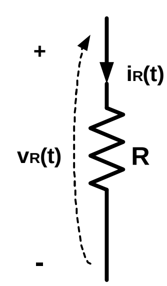
The <u>remaining terms</u> are both **purely sinusoidal** and vary at a frequency that is 2x greater than that of the voltage or current waveforms.

AC Power and Resistors

In the case of a resistive load, whose voltage and current waveforms are:

$$v_R(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

$$i_R(t) = \sqrt{2} \cdot \frac{V}{R} \cdot \sin(\omega \cdot t + \phi)$$



power angle θ equals:

$$\theta = \phi - \delta = \phi - \phi = 0°$$

AC Power and Resistors

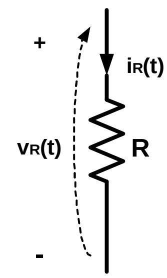
If $\theta=0°$, then the resultant power solution may be simplified to:

$$p_R(t) = V \cdot I - V \cdot I \cdot \cos(2 \cdot \omega \cdot t)$$

since:

$$\cos(\theta) = \cos(0°) = 1$$
$$\sin(\theta) = \sin(0°) = 0$$

This is equivalent the to previous solution for power supplied to a resistive load.



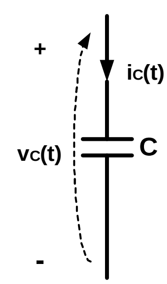
AC Power and Capacitors

Power flowing into a capacitor can be determined from the voltage and current waveforms similar to that for a resistor:

$$p_C(t) = v_C(t) \cdot i_C(t)$$

The solution for the power flowing into a capacitor supplied by an AC source is:

$$p_C(t) = -V \cdot I \cdot \sin(2 \cdot \omega \cdot t)$$



AC Power and Capacitors

If θ = -90° then the general equation

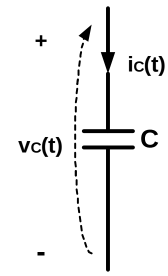
$$p_C(t) = V \cdot I \cdot \cos(\theta)$$
$$- V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t)$$
$$+ V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t)$$

may be simplified to:

$$p_C(t) = -V \cdot I \cdot \sin(2 \cdot \omega \cdot t)$$

since: $\cos(\theta) = \cos(-90°) = 0$

$\sin(\theta) = \sin(-90°) = -1$



Real Power and Capacitors

Since the expression

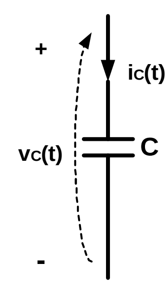
$$p_C(t) = -V \cdot I \cdot \sin(2 \cdot \omega \cdot t)$$

is purely sinusoidal, it has no average value.

Thus, the **Real Power** supplied to the capacitor is **zero**.

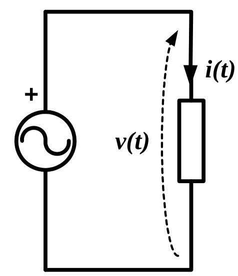
$$P_C = 0 \quad Watts$$

But, there **<u>is</u>** energy being transferred into and out of the capacitor, even if the net energy transfer is zero.



Reactive Power

The term **Reactive Power** is used to characterize the amount of energy that is being temporarily stored and then released by a "reactive" load (capacitive or inductive).



The third term in the general AC power waveform:

$$V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t)$$

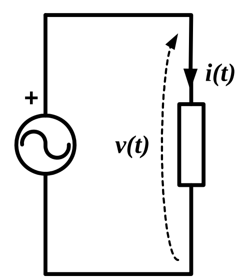
actually relates to this cyclic storage and release of energy by the reactive elements.

Reactive Power

Reactive Power (***Q***) is defined as:

$$Q = V \cdot I \cdot \sin(\theta) \quad Vars$$

which also defines the magnitude of the third term in the power expression.



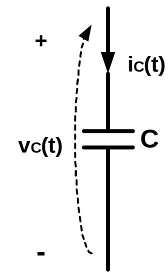
Reactive Power is given the unit of "***Vars***", which stands for:

"***Volt-Amps-Reactive***".

Reactive Power and Capacitors

For a purely capacitive load, reactive power may be solved as:

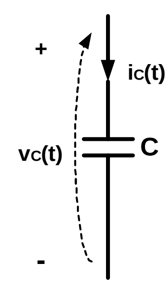
$$
\begin{aligned}
Q_c &= V \cdot I \cdot \sin(\theta) \\
&= V \cdot I \cdot \sin(-90°) \\
&= V \cdot I \cdot (-1) \\
&= -V \cdot I \quad Vars
\end{aligned}
$$



Reactive Power and Capacitors

For a purely capacitive load, reactive power may be solved as:

$$Q_c = -V \cdot I \quad Vars$$



Note that Reactive Power for a capacitor is "**negative**".

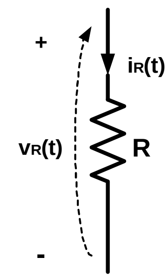
The negative sign relates to the portion of the cycle during which the capacitor is actually storing energy.

Due to the negative sign, capacitors are often characterized as "***producing***" reactive power despite the fact they actually neither consume nor produce a net amount of energy.

Reactive Power and Resistors

Note that for a resistor, reactive power is zero since:

$$Q_R = V \cdot I \cdot \sin(\theta) = V \cdot I \cdot \sin(0°) = 0$$

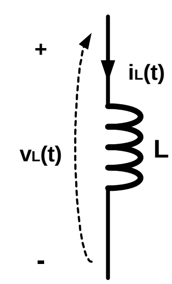


Reactive Power is **only** relates to "reactive" loads… capacitors and inductors, along with the sources that supply the energy they store and release.

AC Power and Inductors

If $\theta = +90°$ then the general equation

$$p_L(t) = V \cdot I \cdot \cos(\theta) - V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t) + V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t)$$



may be simplified to:

$$p_L(t) = +V \cdot I \cdot \sin(2 \cdot \omega \cdot t)$$

since: $\cos(\theta) = \cos(+90°) = 0$

$\sin(\theta) = \sin(+90°) = +1$

Real Power and Inductors

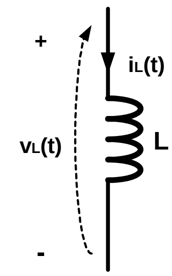
As with capacitors, the solution for power flowing into an inductor:

$$p_L(t) = +V \cdot I \cdot \sin(2 \cdot \omega \cdot t)$$

is purely sinusoidal and has no average value.

Thus, the **Real Power** into the inductor is **zero**.

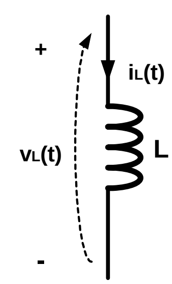
$$P_L = 0 \quad Watts$$



Reactive Power and Inductors

For a purely inductive load, reactive power may be solved as:

$$\begin{aligned} Q_L &= V \cdot I \cdot \sin(\theta) \\ &= V \cdot I \cdot \sin(90°) \\ &= V \cdot I \cdot (+1) \\ &= +V \cdot I \quad Vars \end{aligned}$$



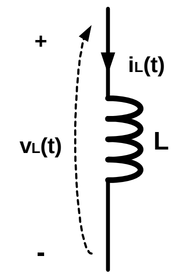
Reactive Power and Inductors

For a purely inductive load, reactive power may be solved as:

$$Q_L = +V \cdot I \quad Vars$$

Note that Reactive Power for an inductor is "**positive**".

Because of the positive sign, inductors are often characterized as "consuming" reactive power



Power in AC Circuits

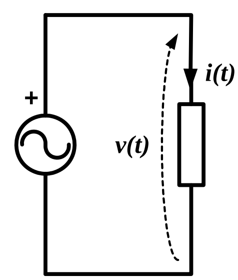
What if the load has both a resistive and a capacitive/inductive component?

(I.e. – a resistor in series with either a capacitor or an inductor)

Then the power angle θ will fall somewhere in the range:

$$-90° \leq \theta \leq +90°$$

resulting in the existence of all three terms in the general power equation.

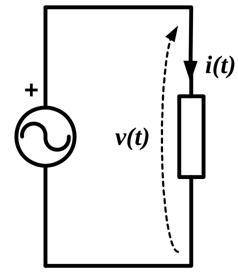


Power in AC Circuits

Thus, for a general load that has both a resistive and a capacitive or an inductive component for which:

$$-90° \leq \theta \leq +90°$$

there will be both Real power and Reactive power flowing into the load, as determined by:



$$P = V \cdot I \cdot \cos(\theta)$$

$$Q = V \cdot I \cdot \sin(\theta)$$

Phasor Representation of Sine Waves

A **phasor** is a representation of a sine-wave whose magnitude, phase and frequency are constant.

Phasors reduce the dependency of these parameters to three independent factors, thus allowing for the simplification of certain types of calculations.

It turns out that, for steady-state AC circuits, the time dependency of the sine-waves can be factored out, reducing the linear differential equations required for their solution to a simpler set of algebraic equations.

Phasors and AC Voltages

The sinusoidal voltage:

$$v(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

may be defined in the form of a ***phasor voltage***:

$$\tilde{V} = Ve^{j\phi} = V\angle\phi$$

such that the voltage is expressed as a complex number in "***polar***" form, having the RMS magnitude V and the phase angle ϕ.

(Note – although the phasor value may be expressed in terms of the "peak" magnitudes, RMS voltage magnitudes are typically used in cases where "power" is of primary interest, and thus will be utilized in this course unless specifically stated otherwise.)

Phasors and AC Voltages

$$v(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

$$\tilde{V} = Ve^{j\phi} = V\angle\phi$$

The phasor voltage defined above is shown both as a complex exponential $Ve^{j\phi}$ and as a complex number in polar form $V\angle\phi$.

Phasors are typically presented in most "circuits" textbooks as complex numbers expressed in "polar" form, but some calculators do not accept this format, thus requiring the use of complex exponentials.

Phasors and AC Voltages

$$v(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

$$\tilde{V} = Ve^{j\phi} = V\angle\phi$$

Note that although $e^{j\phi} = 1\angle\phi$, mathematically the expression $e^{j\phi}$ is only valid if ϕ is defined in radians.

In practice, ϕ is often expressed in degrees, especially when relating to phasors. Some calculators will accept complex exponentials with their angles expressed either in radians or in degrees; other calculators require the angle of the complex exponential to be entered in radians.

Phasors and AC Currents

The sinusoidal current:

$$i(t) = \sqrt{2} \cdot I \cdot \sin(\omega \cdot t + \delta)$$

may also be defined in the form of a ***phasor current***:

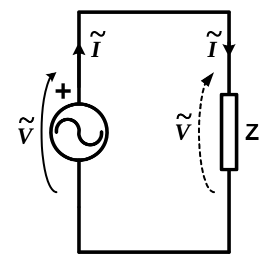
$$\tilde{I} = Ie^{j\delta} = I\angle\delta$$

in which the current is expressed as a complex number in "***polar***" form, having the RMS magnitude I and the phase angle δ.

(Note –RMS current magnitudes will be also utilized in this course unless specifically stated otherwise.)

Impedance

An ***impedance*** is a general expression that is used to define the ratio of the phasor value of the voltage across a load compared to the phasor value of the current flowing through the load.



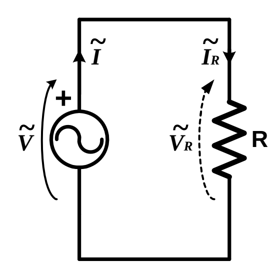
$$Z = \frac{\tilde{V}}{\tilde{I}}$$

Based on this definition, the impedance *Z* can be defined in terms of the voltage and current as:

$$Z = \frac{\tilde{V}}{\tilde{I}} = \frac{V\angle\phi}{I\angle\delta} = \frac{V}{I}\angle(\phi-\delta) = \frac{V}{I}\angle\theta = |Z|\angle\theta$$

Impedance of a Resistor

Since Ohm's Law holds true for resistors given any voltage type, including phasor voltages, the <u>impedance of a resistor</u> is equal to it's resistance:

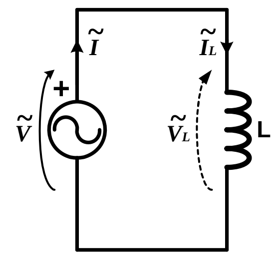


$$Z_R = \frac{\tilde{V}_R}{\tilde{I}_R} = R$$

which is a <u>purely real number</u>.

Impedance of an Inductor

Given the voltage across an inductor, the current flowing through the inductor can be defined, with the following results:



$$v_L(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

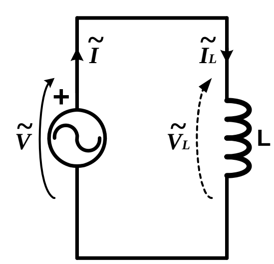
$$i_L(t) = \sqrt{2} \cdot \frac{V}{\omega \cdot L} \cdot \sin(\omega \cdot t + \phi° - 90°)$$

When expressed as phasors, the inductor's voltage and current are:

$$\tilde{V} = V\angle\phi \qquad \tilde{I} = \frac{V}{\omega \cdot L}\angle\phi - 90°$$

Impedance of an Inductor

Based on the inductor's phasor voltage and current:



$$\tilde{V} = V\angle\phi \qquad \tilde{I} = \frac{V}{\omega \cdot L}\angle\phi - 90°$$

the impedance of the inductor can be defined as:

$$Z_L = \frac{\tilde{V}_L}{\tilde{I}_L} = \frac{V\angle\phi}{\left(\frac{V}{\omega \cdot L}\right)\angle\phi - 90°} = (\omega \cdot L)\angle + 90°$$

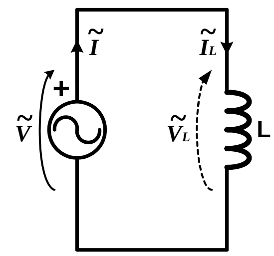
If expressed as a complex number in rectangular form:

$$Z_L = (\omega \cdot L)\angle + 90° = 0 + j\omega \cdot L = j\omega \cdot L$$

Impedance of an Inductor

Based on the inductor's phasor voltage and current:

$$\tilde{V} = V\angle\phi \qquad \tilde{I} = \frac{V}{\omega \cdot L}\angle\phi - 90^\circ$$



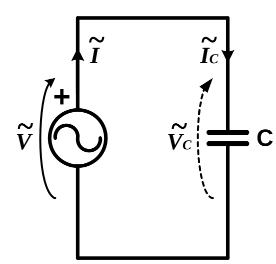
the impedance of the inductor can be defined as:

$$Z_L = (\omega \cdot L)\angle + 90^\circ = j\omega \cdot L$$

which is a <u>positive imaginary number</u>.

Impedance of a Capacitor

Given the voltage across an capacitor, the current flowing through the capacitor can be defined, with the following results:



$$v_C(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

$$i_C(t) = \sqrt{2} \cdot V \cdot \omega \cdot C \cdot \sin(\omega \cdot t + \phi^\circ + 90^\circ)$$

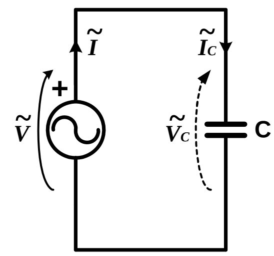
When expressed as phasors, the capacitor's voltage and current are:

$$\tilde{V} = V\angle\phi \qquad \tilde{I} = V \cdot \omega \cdot L\angle\phi + 90^\circ$$

Impedance of a Capacitor

Based on the capacitor's phasor voltage and current:

$$\tilde{V} = V\angle\phi \qquad \tilde{I} = V\cdot\omega\cdot L\angle\phi + 90°$$



the impedance of the capacitor can be defined as:

$$Z_C = \frac{\tilde{V}_C}{\tilde{I}_C} = \frac{V\angle\phi}{V\cdot\omega\cdot C\angle\phi + 90°} = \frac{1}{\omega\cdot C}\angle -90°$$

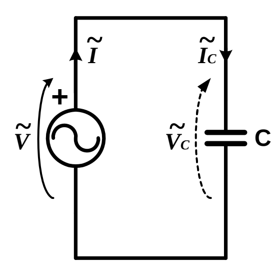
If expressed as a complex number in rectangular form:

$$Z_C = \frac{1}{\omega\cdot C}\angle -90° = 0 - j\frac{1}{\omega\cdot C} = -j\frac{1}{\omega\cdot C}$$

Impedance of a Capacitor

Based on the capacitor's phasor voltage and current:

$$\tilde{V} = V\angle\phi \qquad \tilde{I} = V\cdot\omega\cdot L\angle\phi + 90°$$



the impedance of the capacitor can be defined as:

$$Z_C = \frac{1}{\omega\cdot C}\angle -90° = -j\frac{1}{\omega\cdot C}$$

which is a negative imaginary number.

Reactance

Reactance defines the manner in which capacitive and inductive loads react to a steady-state sinusoidal voltage.

The reactance of a load is equal to the imaginary value of the load's impedance.

Therefore:

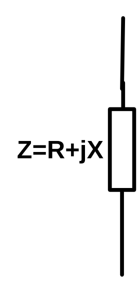
- the reactance of a resistor is $X_R = 0\ \Omega$
- the reactance of an inductor is $X_L = \omega \cdot L\ \Omega$
- the reactance of a capacitor is $X_C = \dfrac{-1}{\omega \cdot C}\ \Omega$

Complex Impedances

A complex impedance ***Z*** is an impedance can have both resistive and reactive (inductive or capacitive) components, and may be expressed in the form:

$$Z = R + jX$$

where: R is the resistive component of the load, and
X is the reactive component of the load.



Note – the impedance of a resistor is $Z_R = R$

– the impedance of an inductor is $Z_L = j(\omega \cdot L)$

– the impedance of a capacitor is $Z_C = -j\left(\dfrac{1}{\omega \cdot C}\right)$

Phasor Analysis of AC Circuits

$$\tilde{V} = V\angle\phi$$

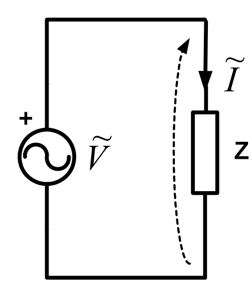
$$\tilde{I} = I\angle\delta$$

When all of the voltages and currents within a steady-state AC circuit are expressed as phasors and all of the circuit elements are defined by their impedance values, the circuit's operation may be solved by a set of algebraic equations based on Ohm's Law.

Phasor Analysis of AC Circuits

If a voltage source having the phasor value $\tilde{V}$ is applied across the complex impedance Z, then the phasor value of the current $\tilde{I}$ may be solved by applying Ohm's Law:

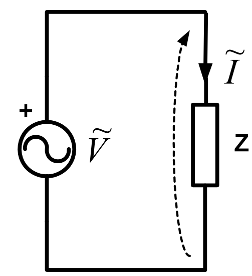
$$\tilde{I} = \frac{\tilde{V}}{Z} = \frac{\tilde{V}}{R + jX} = I\angle\delta$$



Phasor Analysis of AC Circuits

Similarly, given the voltage and current supplied to an impedance:

$$\tilde{V} = V\angle\phi \qquad \tilde{I} = I\angle\delta$$



the impedance may be defined in terms of voltage and current as:

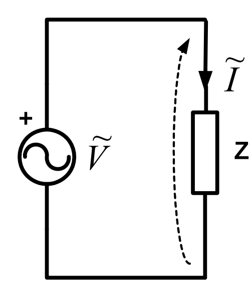
$$Z = \frac{\tilde{V}}{\tilde{I}} = \frac{V\angle\phi}{I\angle\delta} = \frac{V}{I}\angle(\phi-\delta) = \frac{V}{I}\angle\theta = |Z|\angle\theta$$

where Z is a complex number expressed polar-form.

Phasor Analysis of AC Circuits

Thus, given:

$$\tilde{V} = V\angle\phi \qquad \tilde{I} = I\angle\delta$$



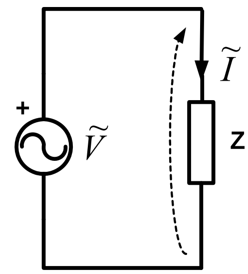
the impedance magnitude is defined by Ohm's Law and the impedance angle is the difference between the voltage and current angles.

$$Z = |Z|\angle\theta \qquad ==> \qquad |Z| = \frac{V}{I} \qquad \theta = \phi - \delta$$

Note that the impedance angle θ is the same as the previously defined "power angle" from the solution for AC power.

Phasor Analysis of AC Circuits

The following formulas may be used to convert an impedance between rectangular form and polar form:



$$Z = |Z|\angle\theta \quad ==> \quad Z = R + jX$$

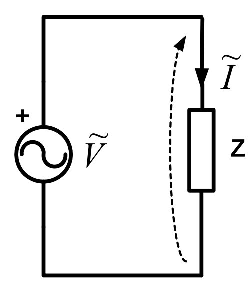
$$\text{R} = |Z| \cdot \cos(\theta) \qquad X = |Z| \cdot \sin(\theta)$$

$$Z = R + jX \quad ==> \quad Z = |Z|\angle\theta$$

$$|Z| = \sqrt{R^2 + X^2} \qquad \theta = \tan^{-1}\frac{X}{R}$$

Phasor Analysis of AC Circuits

Given the voltage and current:



$$\tilde{V} = V\angle\phi \qquad \tilde{I} = I\angle\delta$$

supplied to a complex impedance *Z*, the resultant power angle θ may fall anywhere in the range:

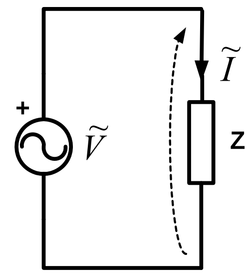
$$-90° \leq \theta \leq +90°$$

resulting in the existence of both a real and a reactive power component being supplied to the load.

Complex Power

The term **Complex Power** is used to characterize both the Real Power and the Reactive Power in an AC system that is being supplied to a complex load impedance that may have a resistive component and/or a reactive component (inductive or capacitive).

$$Z = R + jX$$

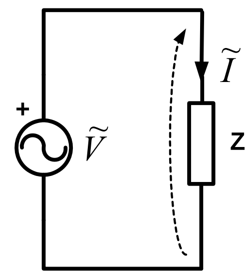


Complex Power

Complex Power (S) is a complex number and is defined by:

$$S = P + j\,Q$$

where: P is Real Power, and
Q is Reactive Power.



Complex Power

Complex Power (S):

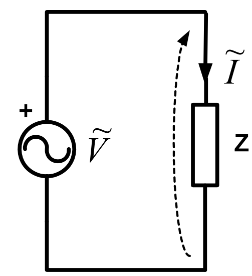
$$S = P + j\,Q$$

may be solved directly from the applied phasor voltage and the resultant phasor current as:

$$S = P + jQ = \tilde{V} \cdot \tilde{I}^{*} = (V\angle\phi)\cdot(I\angle-\delta)$$

$$= V \cdot I \angle(\phi-\delta) = V \cdot I \angle\theta$$

$$= V \cdot I \cdot cos\,\theta + j\,V \cdot I \cdot sin\,\theta$$

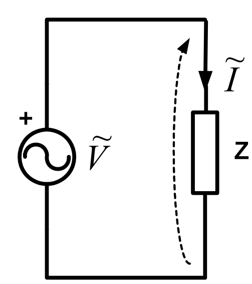


Complex Power

Note that $\tilde{I}^{*}$ is the complex conjugate of $\tilde{I}$ and is defined as:

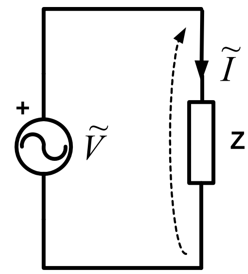
$$\tilde{I}^{*} = (I\angle\delta)^{*} = (I\angle-\delta)$$

In other words, the complex conjugate of a complex number in polar form has the same magnitude as the original number but its angle is negated.



Apparent Power

Apparent Power (|S|) is defined to be the magnitude of complex power:



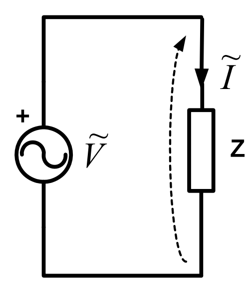
$$|S| = V \cdot I = \sqrt{P^2 + Q^2}$$

Note that apparent power is often specified as part of the ratings of a machine such that:

$$|S|_{rated} = V_{rated} \cdot I_{rated}$$

Power Factor

Power Factor (*pf*) provides a measure of the portion of apparent power that actually relates to real power:



$$pf = \frac{P}{|S|}$$

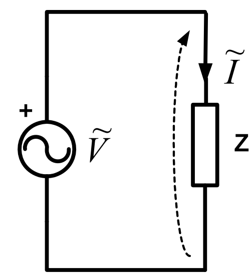
Thus, power factor may be solved as:

$$pf = \frac{P}{|S|} = \frac{V \cdot I \cdot cos\,\theta}{V \cdot I} = cos\,\theta$$

Power Factor

Power Factor is often characterized by a qualifier, either **leading** or **lagging**.

This qualifier describes the phase angle relationship between a phasor voltage and it's associated phasor current.

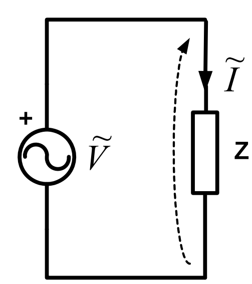


Power Factor

A **leading** power factor exists when the current is "leading" the voltage, which occurs when the load impedance has a **capacitive** component, resulting in a negative angle difference for θ:

$$\theta = \phi - \delta$$

$$-90° \le \theta < 0°$$

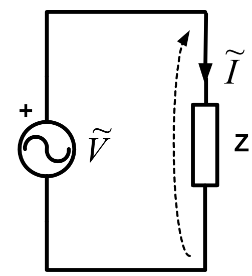


Power Factor

A **lagging** power factor exists when the current is "lagging" the voltage, which occurs when the load impedance has an **inductive** component, resulting in a positive angle difference for θ:

$$\theta = \phi - \delta$$

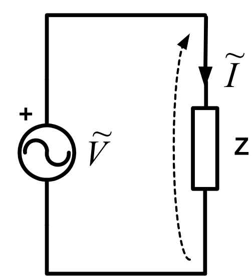
$$0° < \theta \leq +90°$$



Power Factor

Note that a purely resistive load in a zero value for the angle θ, which is neither leading nor lagging. This is often defined as a **unity** power factor since the value of power factor under this condition equals one.

$$\cos(\theta) = \cos(0°) = 1$$



Summary of Complex Power Equations

Complex Power (S): $S = P + jQ = \tilde{V} \cdot \tilde{I}^*$

Real Power (P): $P = V \cdot I \cdot \cos\theta$

Reactive Power (Q): $Q = V \cdot I \cdot \sin\theta$

Apparent Power (|S|): $|S| = V \cdot I = \sqrt{P^2 + Q^2}$

Power Factor (pf): $pf = \cos\theta$