Complex Power

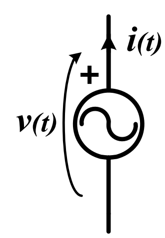
in

Steady-State AC Circuits

ECET 3500 – *Survey of Electric Machines*

Steady-State AC Voltage Sources

The voltage potential produced by an AC source may be defined as:



$$v(t) = V_{peak} \cdot \sin(\omega \cdot t + \phi)$$

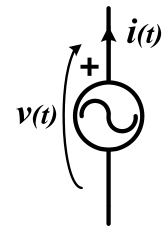
where: V_{peak} is the peak value of the voltage,
ω is the angular frequency ($2\pi f$) of the waveform, and
ϕ is the phase angle of the voltage waveform.

Steady-State AC Voltage Sources

Similarly, the current produced by the AC source may be defined as:

$$i(t) = I_{peak} \cdot \sin(\omega \cdot t + \delta)$$

where: I_{peak} is the peak value of the current,
ω is the angular frequency ($2\pi f$) of the waveform, and
δ is the phase angle of the current waveform.



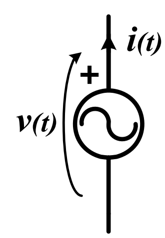
Power in AC Circuits

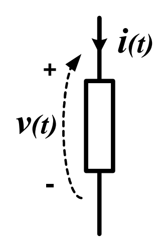
<u>Electric Power</u> is defined as the **rate** at which **electric energy** is either produced or consumed by an element within the circuit.

The **power** produced or consumed by a circuit element can be determined from the voltage and current waveforms associated with that element:

$$p(t) = v(t) \cdot i(t) \quad \text{(Watts)}$$

where: $p(t)$ is the instantaneous rate that a circuit element either produces or consumes energy at time t.





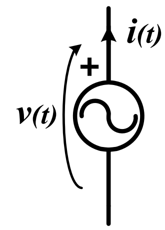
Source vs. Load Convention

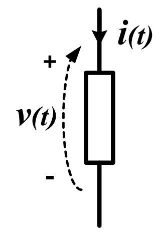
Note that the expression:

$$p(t) = v(t) \cdot i(t)$$

defines the **power** "**PRODUCED**" by an element when the current is defined in the same direction as the voltage-rise across the element.

But, if the current is defined in the opposite direction as the voltage-rise across an element, then $p(t)$ defines the **power** "**CONSUMED**" by that element.





Power from an AC Source

In the case of an AC source where:

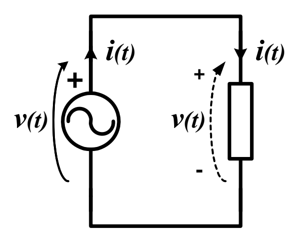
$$v(t) = V_{peak} \cdot \sin(\omega \cdot t + \phi)$$

$$i(t) = I_{peak} \cdot \sin(\omega \cdot t + \delta)$$

the general expression for power produced by the source is:

$$p(t) = V_{peak} \cdot I_{peak} \cdot \sin(\omega \cdot t + \phi) \cdot \sin(\omega \cdot t + \delta)$$

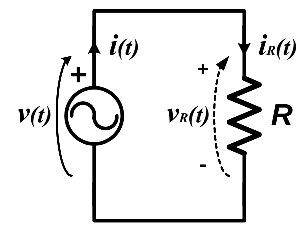
To better understand the nature of this expression, it may by useful to first consider the case where the voltage source is supplying a resistive load.



AC Sources and Resistive Loads

Given the voltage, v_R, across a **resistive load**:

$$v_R(t) = V_{peak} \cdot \sin(\omega \cdot t)$$



the resultant current, i_R, that flows through the resistor is defined by Ohm's Law as:

$$i_R(t) = I_{peak} \cdot \sin(\omega \cdot t + \delta)$$

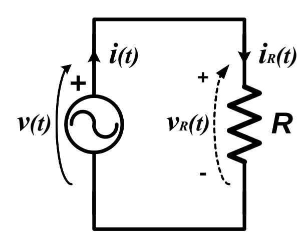
$$= \frac{v_R(t)}{R}$$

$$= \frac{V_{peak}}{R} \cdot \sin(\omega \cdot t)$$

AC Sources and Resistive Loads

Thus, for a **resistive load**:

$$v_R(t) = V_{peak} \cdot \sin(\omega \cdot t) \qquad i_R(t) = \frac{V_{peak}}{R} \cdot \sin(\omega \cdot t)$$



the **peak value** of the current also adheres to **Ohm's Law**:

$$I_{peak} = \frac{V_{peak}}{R}$$

and the **phase angle** of the current equals the phase angle of the applied voltage…

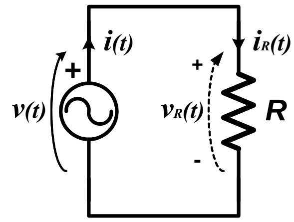
$$\delta = \phi = 0°$$

There is no phase shift between the voltage and current waveforms relating to a purely resistive load.

AC Power and Resistors

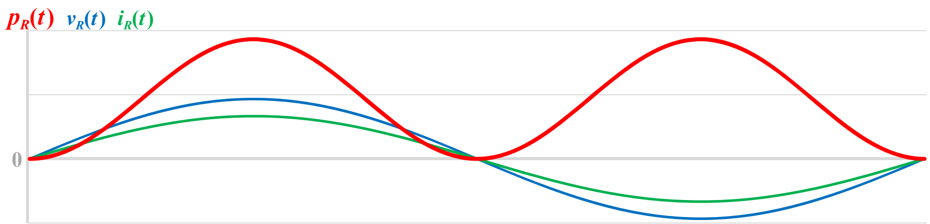
Given a resistor's voltage and current waveforms:

$$v_R(t) = V_{peak} \cdot \sin(\omega \cdot t) \qquad i_R(t) = \frac{V_{peak}}{R} \cdot \sin(\omega \cdot t)$$



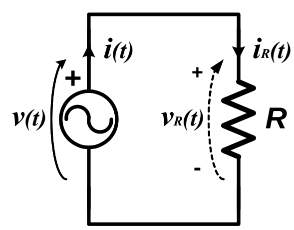
the **power** consumed by the resistor is:

$$p_R(t) = \frac{V_{peak}^2}{R} \cdot \sin^2(\omega \cdot t)$$

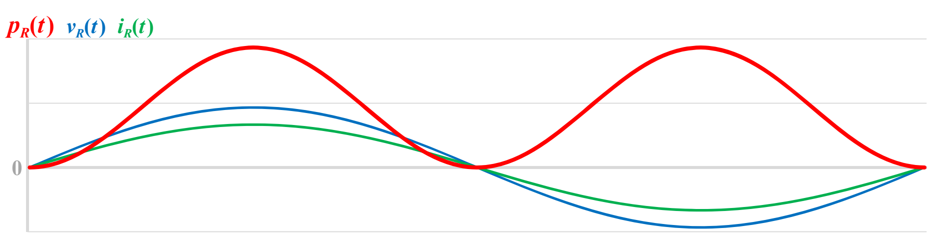


AC Power and Resistors

Note that the resistor **power** varies periodically but at a frequency that is 2x larger than that of the resistor's voltage and current waveforms.



Also note that the **power** consumed by the resistor is always non-negative, which is expected since any negative power values would imply that the resistor is instantaneously "producing" power.

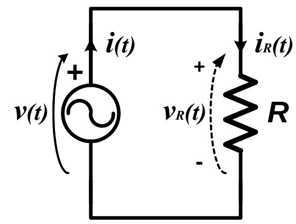


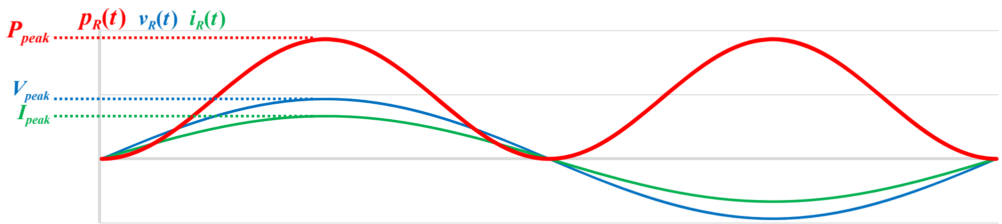
AC Power and Resistors

Although the **power** consumed by a resistor varies periodically between zero and its peak value under steady-state conditions:

$$P_{peak} = V_{peak} \cdot I_{peak}$$

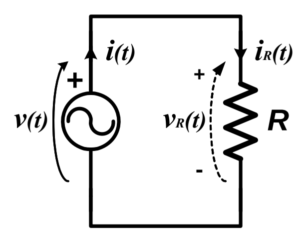
a resistor's operation is often characterized in terms of the average power that it consumes.





AC Power and Resistors

To better understand the **resistor power waveform**, it is useful to rewrite the power equation into the following form:



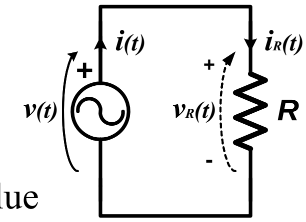
$$\begin{aligned} p_R(t) &= V_{peak} \cdot I_{peak} \cdot \sin^2(\omega \cdot t) \\ &= \frac{V_{peak} \cdot I_{peak}}{2} \cdot [1 - \cos(2 \cdot \omega \cdot t)] \\ &= \boxed{\frac{V_{peak} \cdot I_{peak}}{2} - \frac{V_{peak} \cdot I_{peak}}{2} \cdot \cos(2 \cdot \omega \cdot t)} \end{aligned}$$

(by utilizing the trigonometric identity $\sin^2 x = ½ \cdot [1 - \cos 2x]$)

AC Power and Resistors

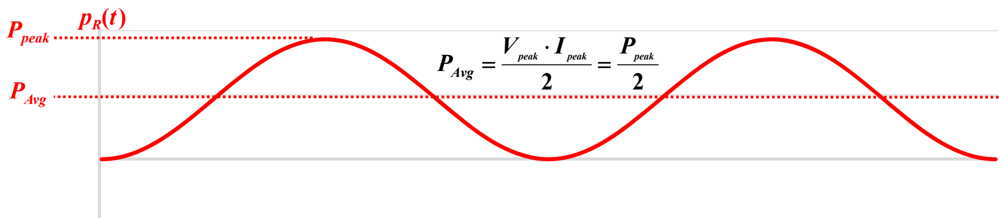
Looking at the resultant **resistor power waveform**:

$$p_R(t) = \boxed{\frac{V_{peak} \cdot I_{peak}}{2}} - \frac{V_{peak} \cdot I_{peak}}{2} \cdot \cos(2 \cdot \omega \cdot t)$$



It can be seen that the expression has <u>two terms</u>:

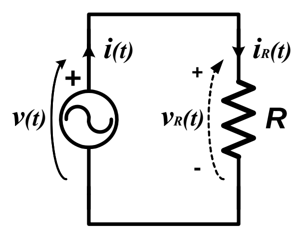
- The **first term** is a **constant** that provides the value of the <u>average power</u> consumed by the resistor.



AC Power and Resistors

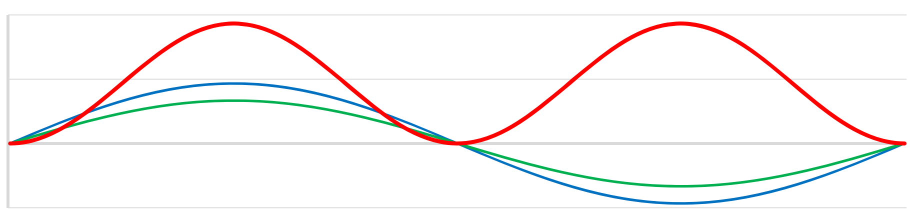
Looking at the resultant **resistor power waveform**:

$$p_R(t) = \frac{V_{peak} \cdot I_{peak}}{2} - \boxed{\frac{V_{peak} \cdot I_{peak}}{2} \cdot \cos(2 \cdot \omega \cdot t)}$$



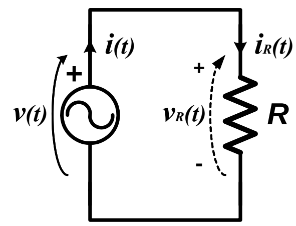
It can be seen that the expression has <u>two terms</u>:

- The **second term** is a **purely "sinusoidal"** term that has a <u>zero average value</u> and varies with a <u>frequency that is 2x larger than that of the source</u>.



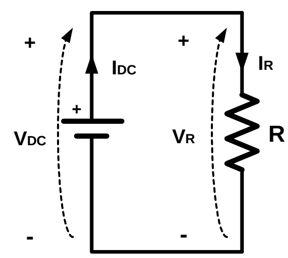
Real Power

In an AC system, the **Real Power**, ***P***, produced or consumed by a circuit element is defined in terms of the **average power** produced or consumed by that element:



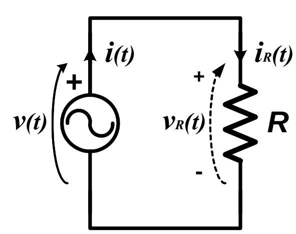
$$P = Avg[p(t)] = Avg[v(t) \cdot i(t)] \quad (Watts)$$

Note – since power is constant in a DC circuit, this definition for real power also applies to the power produced or consumed by any element in a DC circuit.



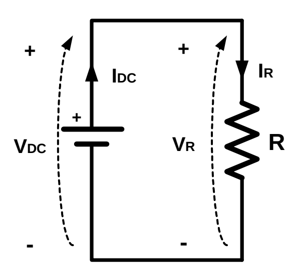
AC vs. DC Power in Resistors

Given a resistor supplied by an AC source, the **real power**, $P_{R(AC)}$, consumed by the resistor is the average value of the its power waveform, which is only ½ that of its peak value:



$$P_{R(AC)} = \frac{P_{peak}}{2} = \frac{V_{peak} \cdot I_{peak}}{2} \quad (Watts)$$

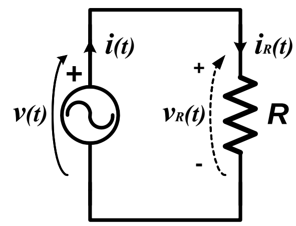
Yet, this result may cause confusion if the AC real power value is compared to the constant power supplied to a resistor by a DC source.

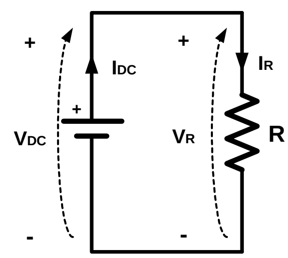


AC vs. DC Power in Resistors

If the peak value of the AC source is equal to the magnitude of a separate DC source ($V_{peak} = V_{DC}$) and both sources supply similar resistors, then:

the real power consumed by the AC-supplied resistor will only be ½ that of the power consumed by the DC-supplied resistor .





$$P_{R(AC)} = \frac{V_{peak} \cdot I_{peak}}{2} = \frac{V_{peak}^2}{2 \cdot R} \quad (Watts)$$

$$P_{R(DC)} = V_{DC} \cdot I_{DC} = \frac{V_{DC}^2}{R} \quad (Watts)$$

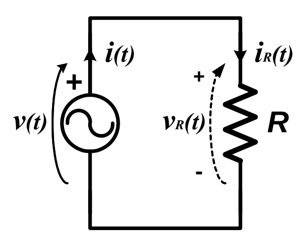
AC vs. DC Power in Resistors

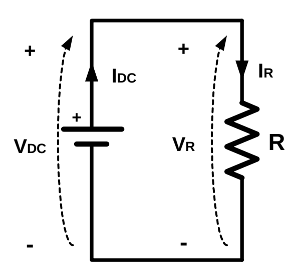
In other words:

Given an <u>AC source whose peak value is equal to the magnitude of a DC source</u>,

If <u>both sources supply similar resistors</u>,

Then **the AC source will be ½ as <u>effective</u> as the DC source** in terms of power supplied to a resistor.





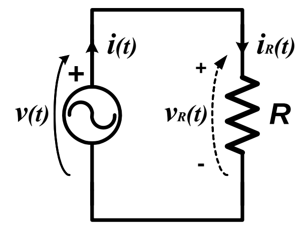
If $V_{peak} = V_{DC}$ → $P_{R(AC)} = \frac{P_{R(DC)}}{2}$ (Watts)

AC vs. DC Power in Resistors

For example:

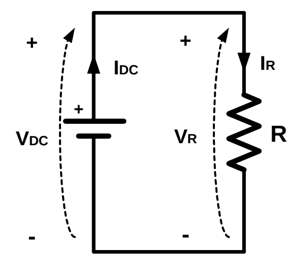
The **real power** consumed by a **500Ω** resistor supplied by a **100V$_{\text{peak}}$ AC source** is:

$$P_{R(AC)} = \frac{V_{peak}^2}{2 \cdot R} = \frac{100^2}{2 \cdot 500} = 10\,Watts$$



And the **power** consumed by a **500Ω** resistor supplied by a **100V DC source** is:

$$P_{R(DC)} = \frac{V_{DC}^2}{R} = \frac{100^2}{500} = 20\,Watts$$

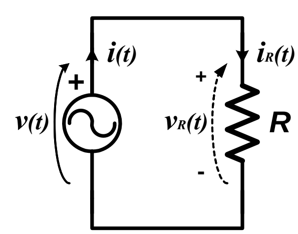


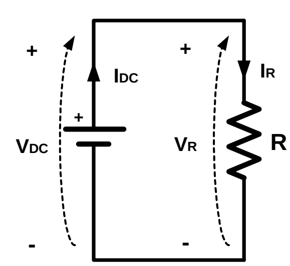
Effective Voltage

But, given an AC source with peak voltage V_{peak},

If an **<u>effective voltage</u> magnitude**, V_{eff}, is defined for the AC source in terms of the magnitude of a DC source that would deliver the same average power to a resistor, then:

"What is the effective voltage magnitude of an AC source with peak voltage V_{peak}?"





Effective Voltage

Given:

$$P_{R(AC)} = \frac{V_{peak}^2}{2 \cdot R} \qquad P_{R(DC)} = \frac{V_{DC}^2}{R}$$

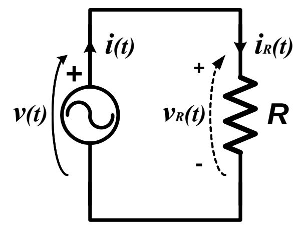
if:

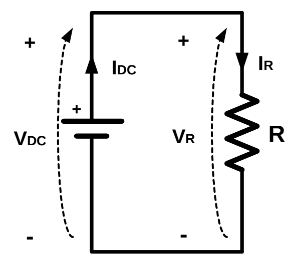
$$V_{DC} = \frac{V_{peak}}{\sqrt{2}}$$

then the AC and DC sources will both supply **equal average (real) powers** to the resistor.

Thus, the **<u>effective voltage</u>** of the AC source is:

$$V_{effective} = \frac{V_{peak}}{\sqrt{2}}$$





Effective Voltage

<u>For example</u>:

Given: $V_{peak} = 100\text{V}$ and $R = 500\Omega$,

$$P_{R(AC)} = \frac{V_{peak}^2}{2 \cdot R} = \frac{100^2}{2 \cdot 500} = 10 \quad \text{(Watts)}$$

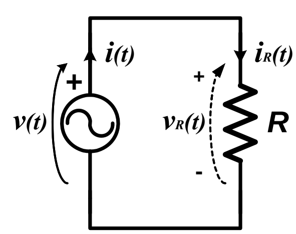
but:

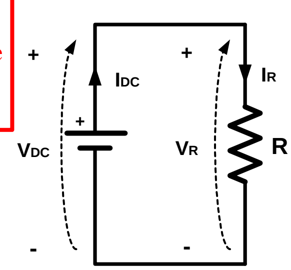
$$V_{effective} = \frac{V_{peak}}{\sqrt{2}} = \frac{100}{\sqrt{2}} = 70.7\text{V}$$

if: $V_{DC} = 70.7\text{V}$ and $R = 500\Omega$:

$$P_{R(DC)} = \frac{V_{DC}^2}{R} = \frac{70.7^2}{500} = 10 \quad \text{(Watts)}$$

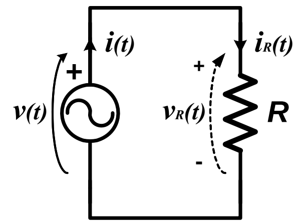
The 100V_{peak} AC source is just as effective as the 70.7V DC source





RMS Voltage Magnitude

It turns out that the **effective voltage** magnitude of a sinusoidal AC source:



$$v(t) = V_{peak} \cdot \sin(\omega \cdot t + \phi) \qquad V_{effective} = \frac{V_{peak}}{\sqrt{2}}$$

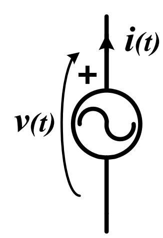
is equal to the **RMS** (***root-mean-squared***) value of its AC waveform, as defined by the function:

$$V_{eff} = V_{RMS} = \sqrt{\frac{1}{T} \cdot \int_0^T v^2(t) \cdot dt}$$

Note – this function can also be used to determine the effective magnitude of any arbitrary periodic (AC) waveform.

RMS Magnitudes

The voltage potential and current produced by an AC source can also be expressed in terms of their **RMS magnitudes**, ***V*** and ***I*** respectively:



$$v(t) = V_{peak} \cdot \sin(\omega \cdot t + \phi) = \boxed{\sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)}$$

$$i(t) = I_{peak} \cdot \sin(\omega \cdot t + \delta) = \boxed{\sqrt{2} \cdot I \cdot \sin(\omega \cdot t + \delta)}$$

where: $V = \frac{V_{peak}}{\sqrt{2}}$ is the RMS magnitude of the AC voltage, and

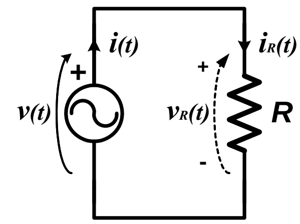
$I = \frac{I_{peak}}{\sqrt{2}}$ is the RMS magnitude of the AC current.

RMS Magnitudes & Resistor Power

When expressed in terms of their RMS magnitudes:

$$v(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t)$$

$$i(t) = \sqrt{2} \cdot I \cdot \sin(\omega \cdot t)$$



the **power** delivered to a resistor is:

$$p_R(t) = V \cdot I - V \cdot I \cdot \cos(2 \cdot \omega \cdot t)$$

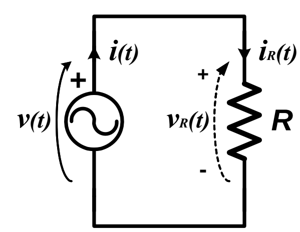
which has an (average) **Real Power** value of:

$$P_{R(AC)} = V \cdot I = \frac{V^2}{R}$$

RMS Magnitudes & Resistor Power

The result:

$$P_{R(AC)} = V \cdot I = \frac{V^2}{R}$$



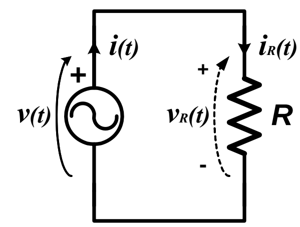
is similar to the DC formula for power:

$$P_{R(DC)} = V_{DC} \cdot I_{DC} = \frac{V_{DC}^2}{R}$$

which provides an advantage for defining the AC waveforms in terms of their RMS magnitudes instead of their peak values.

Real Power and Resistors

Real Power (***P***) is the average power produced or consumed by an element in an AC circuit.



Real Power is defined in units of ***Watts***.

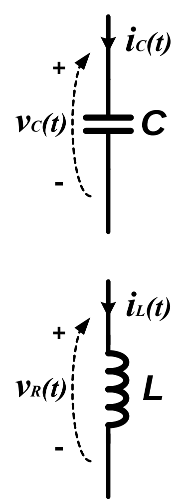
In a **purely resistive** AC circuit, if the voltages and currents are expressed in terms of their RMS magnitudes, then *real power* can be calculated as:

$$P = V \cdot I \quad Watts$$

AC Sources and Reactive Loads

What if the AC source is supplying a load that is **purely reactive**…

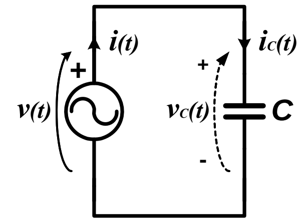
I.e. – either **Capacitive** or **Inductive**?



Although a sinusoidal (AC) voltage source will cause a sinusoidal (AC) current to flow through both capacitors and inductors, their voltage and current waveforms do not follow the linear Ohm's Law relationship. Instead, their voltage and current waveforms are governed by a differential relationship.

AC Sources and Capacitors

For an ideal **capacitor**, the voltage-current relationship is defined by the following equations:



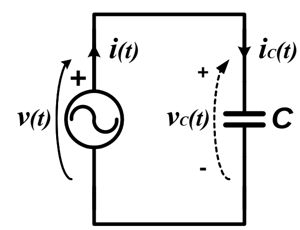
$$i_C(t) = C \cdot \frac{dv_C(t)}{dt}$$

$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(t)dt = \frac{1}{C}\int_{0}^{t} i_C(t)dt + V_o$$

We may obtain a solution for steady-state AC operation from these relationships.

AC Sources and Capacitors

Given the voltage applied across a **capacitor**:



$$v_C(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t)$$

the resultant current will be:

$$i_C(t) = \sqrt{2} \cdot V \cdot \omega \cdot C \cdot \sin(\omega \cdot t + 90°)$$

Note that:

- The capacitor current is phase-shifted by +90° compared to the capacitor voltage, and
- The voltage and current magnitudes do not follow a linear relationship w.r.t. capacitance.

$$V = I \cdot \frac{1}{\omega \cdot C}$$

AC Power and Capacitors

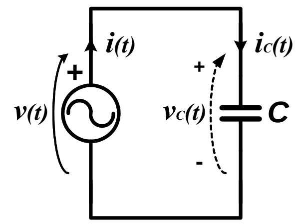
Given a capacitor's voltage and current waveforms:

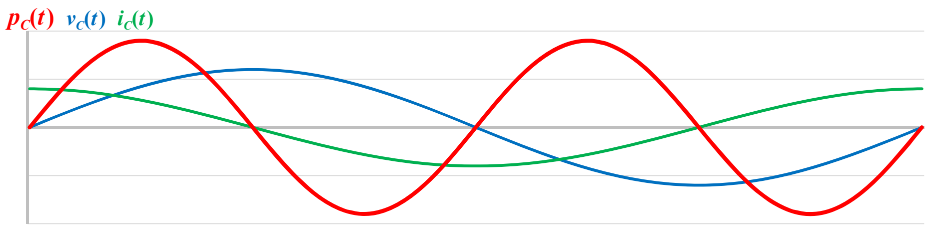
$$v_C(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t)$$

$$i_C(t) = \sqrt{2} \cdot V \cdot \omega \cdot C \cdot \sin(\omega \cdot t + 90°)$$

the **power** consumed by the capacitor is:

$$p_C(t) = V^2 \cdot \omega \cdot C \cdot \sin(2 \cdot \omega \cdot t)$$





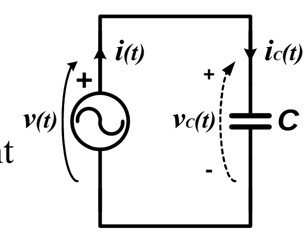
AC Power and Capacitors

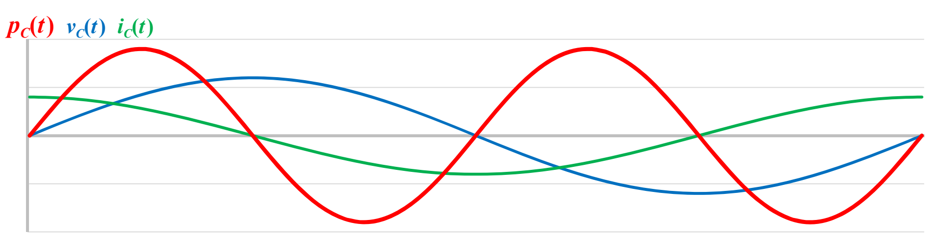
Looking at the resultant **capacitor power waveform**:

$$p_C(t) = V^2 \cdot \omega \cdot C \cdot \sin(2 \cdot \omega \cdot t)$$

it can be seen that it varies sinusoidally at twice the frequency of the capacitor's voltage and current waveforms and that it has a **zero-average value**.

$$P_C = 0$$ ⬅ **(zero real power)**



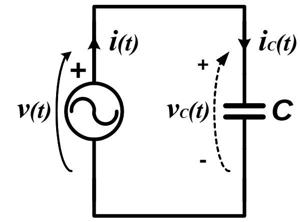


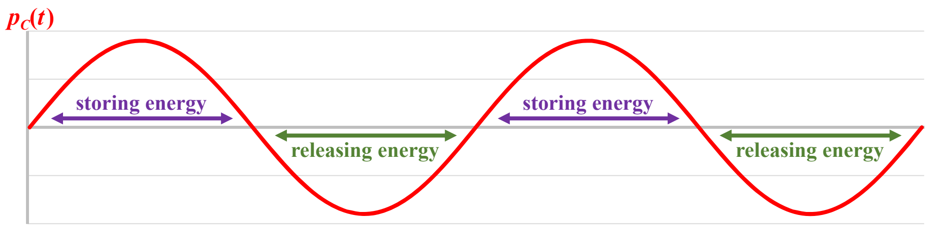
AC Power and Capacitors

Despite the fact that the (average) real power consumed by a capacitor is zero:

$$P_C = 0$$

there is energy flowing into and out of the capacitor as it temporarily stores and releases a charge in a periodic manner.



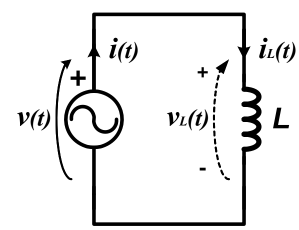


AC Sources and Inductors

For an ideal **inductor**, the voltage-current relationship is defined by the following equations:

$$v_L(t) = L \cdot \frac{di_L(t)}{dt}$$

$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(t)dt = \frac{1}{L}\int_{0}^{t} v_L(t)dt + I_o$$



We may obtain a solution for steady-state AC operation from these relationships.

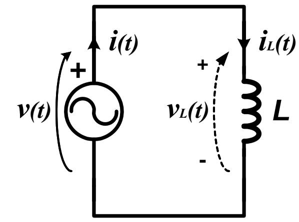
AC Sources and Inductors

Given the voltage applied across a **inductor** :

$$v_L(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t)$$

the resultant current will be:

$$i_L(t) = \sqrt{2} \cdot \frac{V}{\omega \cdot L} \cdot \sin(\omega \cdot t - 90°)$$



Note that:

- The inductor current is phase-shifted by –90° compared to the inductor voltage, and
- The voltage and current magnitudes do not follow a linear relationship w.r.t. inductance.

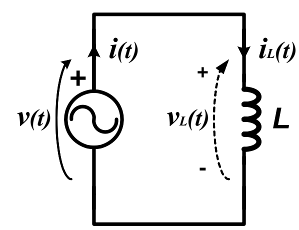
$$V = I \cdot \omega \cdot L$$

AC Power and Inductors

Given an **inductor's** voltage and current waveforms:

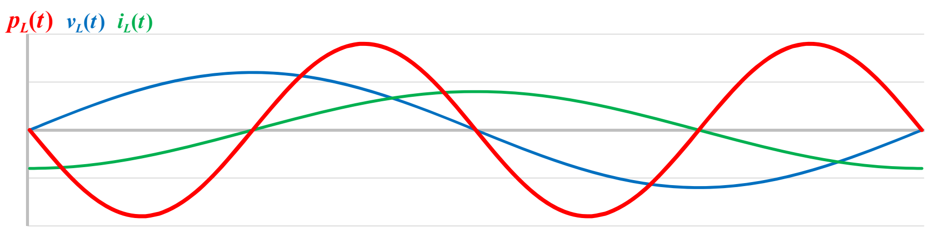
$$v_L(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t)$$

$$i_L(t) = \sqrt{2} \cdot \frac{V}{\omega \cdot L} \cdot \sin(\omega \cdot t - 90°)$$



the **power** consumed by the inductor is:

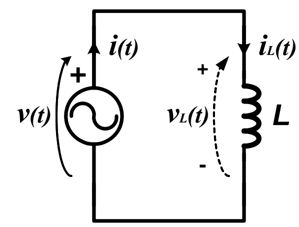
$$p_L(t) = -\frac{V^2}{\omega \cdot L} \cdot \sin(2 \cdot \omega \cdot t)$$



AC Power and Inductors

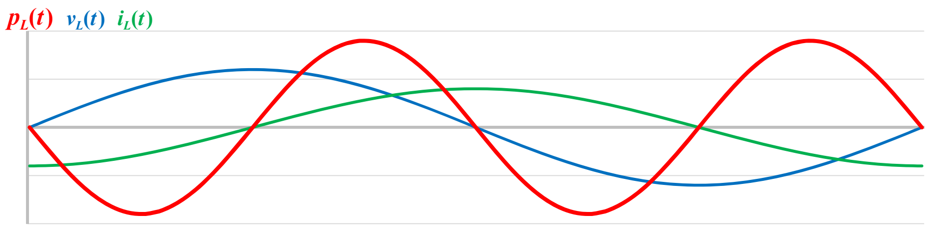
Looking at the resultant **inductor power waveform**:

$$p_L(t) = -\frac{V^2}{\omega \cdot L} \cdot \sin(2 \cdot \omega \cdot t)$$



it can be seen that it varies sinusoidally at twice the frequency of the inductor's voltage and current waveforms and that it has a **zero-average value**.

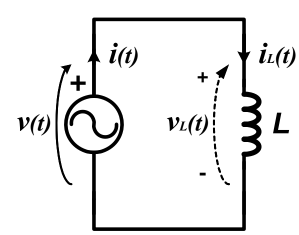
$$P_L = 0$$ ⬅(zero real power)



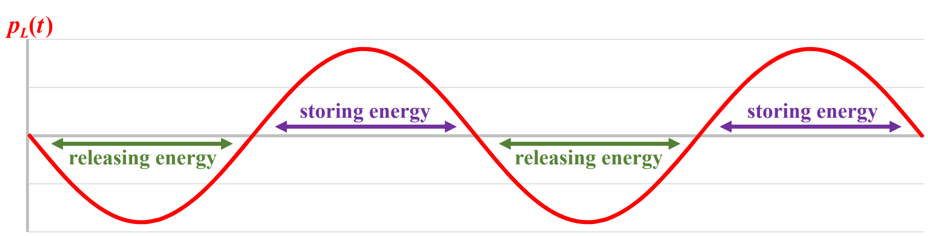
AC Power and Inductors

Despite the fact that the (average) real power consumed by an inductor is zero:

$$P_L = 0$$

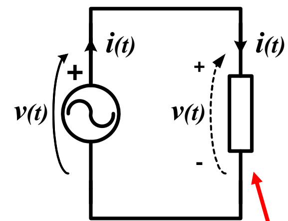


the inductor also temporarily stores and releases energy in a periodic manner.



AC Power – General Case

Given a source's voltage and current waveforms expressed in terms of their RMS magnitudes:



$$v(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

$$i(t) = \sqrt{2} \cdot I \cdot \sin(\omega \cdot t + \delta)$$

the general expression for the instantaneous **power** produced by the AC source is:

$$p(t) = v(t) \cdot i(t)$$

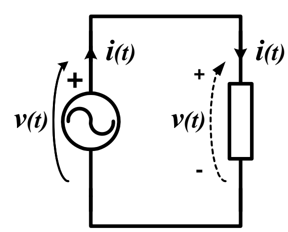
$$= 2 \cdot V \cdot I \cdot \sin(\omega \cdot t + \phi) \cdot \sin(\omega \cdot t + \delta)$$

Note that this expression also defines the power consumed by the load

AC Power – General Case

The instantaneous **power** expression:

$$p(t) = 2 \cdot V \cdot I \cdot \sin(\omega \cdot t + \phi) \cdot \sin(\omega \cdot t + \delta)$$



can be modified using several trigonometric identities into the following form:

$$p(t) = V \cdot I \cdot \cos(\theta) - V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t) - V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t)$$

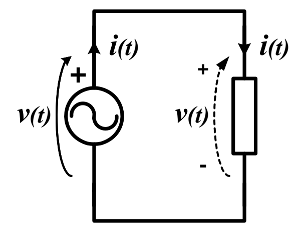
where:

$\theta = \phi - \delta$ *(angle of the voltage minus angle of the current)*

AC Power – General Case

The expression for **AC power** has three terms:

$$p(t) = \boxed{V \cdot I \cdot \cos(\theta)}$$
$$- V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t)$$
$$- V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t)$$



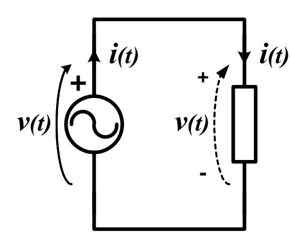
The first term is a constant that provides the average or **real power** produced by the source:

$$P = V \cdot I \cdot \cos(\theta)$$

AC Power – General Case

The expression for **AC power** has three terms:

$$p(t) = V \cdot I \cdot \cos(\theta)$$
$$\boxed{\begin{aligned} &- V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t) \\ &- V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t) \end{aligned}}$$

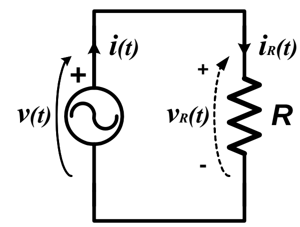


The remaining terms are both purely sinusoidal and vary at a frequency that is 2x greater than that of the voltage or current waveforms.

AC Power and Resistors

For a **purely resistive** load ($\theta = 0°$), the general **power** waveform:

$$p(t) = V \cdot I \cdot \cos(\theta) - V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t) - V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t)$$



simplifies to:

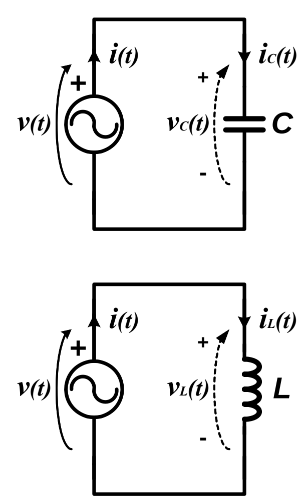
$$p(t) = V \cdot I - V \cdot I \cdot \cos(2 \cdot \omega \cdot t)$$

which is equivalent to the previously determined result for a purely resistive load.

AC Power and Capacitors or Inductors

For a **purely capacitive** load ($\theta = -90°$) or a **purely inductive** load ($\theta = +90°$), the waveform:

$$p(t) = V \cdot I \cdot \cos(\theta) - V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t) - V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t)$$



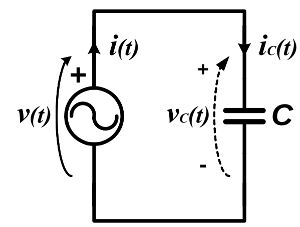
simplifies to:

$$p(t) = \pm V \cdot I \cdot \sin(2 \cdot \omega \cdot t)$$

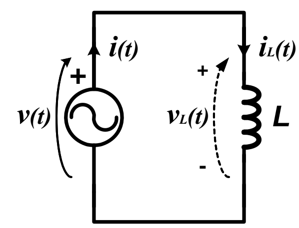
which is similar to the previously determined results for capacitive and inductive loads.

Reactive Power

The term **<u>Reactive Power</u>** is used to characterize the amount of energy that is temporarily stored and released by **reactive loads**. (I.e. – capacitive or inductive loads).



Reactive Power, ***Q***, is defined as the peak value of the power that flows in and out of a reactive load.



Since, for a purely reactive load:

$$p(t) = \pm V \cdot I \cdot \sin(2 \cdot \omega \cdot t)$$

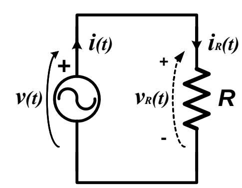
then:

$$Q_C = V \cdot I \quad \text{or} \quad Q_L = V \cdot I \quad (VARs)$$

VARs ≡ Volt·Amps·Reactive

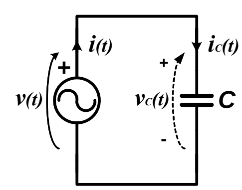
Real Power and Reactive Power

Although a voltage source can produce (or consume) both real power and reactive power, in terms of **passive loads** (resistors, capacitors and inductors):



Real power is consumed only by <u>resistive loads</u>.

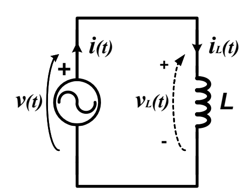
(Ideal capacitors and inductors consume zero-average power)



$$P_C = 0 \qquad P_L = 0$$

Reactive power only relates to <u>reactive loads</u>.

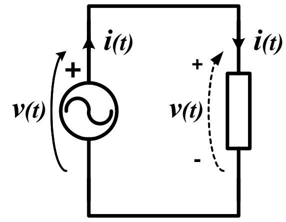
(Resistors do not temporarily store and release energy)



$$Q_R = 0$$

AC Power in Mixed (R-L-C) Circuits

If a source is connected to a circuit that contains a combination of resistive, capacitive, or inductive loads, then the angle difference, θ, between the phase angles of the voltage and current will be:



$$-90° \le \theta \le +90°$$

In a circuit with <u>multiple load-types</u>, the angle difference:

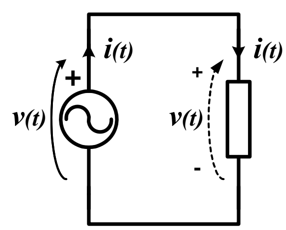
$$\theta \ne 0°, -90°, \text{ or } +90°$$

thus <u>all three terms will exist</u> in the power waveform:

$$p(t) = V \cdot I \cdot \cos(\theta)$$
$$- V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t)$$
$$- V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t)$$

Real Power in R-L-C Circuits

Given a source that is connected to a circuit that contains <u>multiple load-types</u> and the general power waveform:



$$p(t) = \boxed{V \cdot I \cdot \cos(\theta)}$$
$$- V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t)$$
$$- V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t)$$

$$v(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$
$$i(t) = \sqrt{2} \cdot I \cdot \sin(\omega \cdot t + \delta)$$
$$\theta = \phi - \delta$$
$$-90° \le \theta \le +90°$$

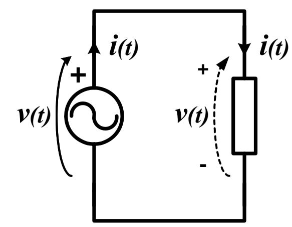
Real power is defined as the average value of the power waveform:

$$P = V \cdot I \cdot \cos(\theta) \quad \textit{Watts}$$

Reactive Power in R-L-C Circuits

Given a source that is connected to a circuit that contains multiple load-types and the general power waveform:

$$p(t) = V \cdot I \cdot \cos(\theta)$$
$$- V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t)$$
$$- \boxed{V \cdot I \cdot \sin(\theta)} \cdot \sin(2 \cdot \omega \cdot t)$$



$$v(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$
$$i(t) = \sqrt{2} \cdot I \cdot \sin(\omega \cdot t + \delta)$$
$$\theta = \phi - \delta$$
$$-90° \leq \theta \leq +90°$$

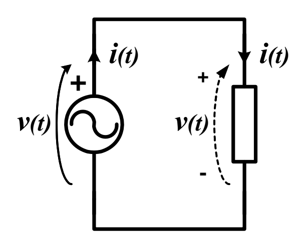
Reactive power is magnitude of the third term which relates to the power that flows in and out of a reactive elements in the circuit:

$$Q = V \cdot I \cdot \sin(\theta) \quad VARs$$

Reactive Power – Capacitors vs. Inductors

Although **Reactive Power**, Q, is defined as the magnitude of the power that flows in and out of the reactive loads:

$$Q = V \cdot I \cdot \sin(\theta)$$



The reactive power equation will return a **negative value** for a circuit that is primarily **capacitive** (**–90° ≤ θ ≤ 0°**), and

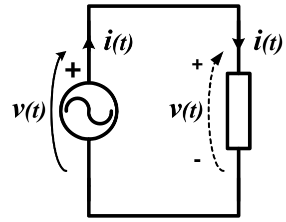
The reactive power equation will return a **positive value** for a circuit that is primarily **inductive** (**–90° ≤ θ ≤ 0°**).

Despite being defined as a magnitude, the sign is often included to characterize the type of the load (capacitive or inductive) to which the reactive power relates.

Due to the resultant signs, capacitors are often characterized as "producing" reactive power and inductors are often characterized as "consuming" reactive power despite neither actually consuming or producing a net amount of energy

Phasor Analysis of an AC Circuit

When performing a **phasor analysis** on an AC circuit, the sinusoidal voltages and currents:



$$v(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

$$i(t) = \sqrt{2} \cdot I \cdot \sin(\omega \cdot t + \delta)$$

are defined by their **phasor values**:

$$\tilde{V} = Ve^{j\phi} = V\angle\phi$$

$$\tilde{I} = Ie^{j\delta} = I\angle\delta$$

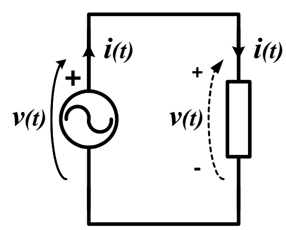
The phasor values are also shown in complex exponential form because some calculators do not allow complex numbers to be expressed in polar form.

When expressed in polar form, the angles may be defined either in degrees or radians. But, when expressed in complex exponential form, most calculators require the angles to be defined in radians.

such that they are expressed as **complex numbers** in **polar form** with the RMS magnitude and phase angle of their respective sinusoidal waveform.

Phasor Analysis of an AC Circuit

Additionally, when performing a **phasor analysis** on an AC circuit, the individual load elements:



R, ***L***, and ***C***

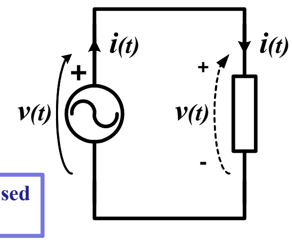
are defined in terms of their **impedance values**, ***Z***, such that for:

Circuit Elements		Impedance Values
Resistors	➡	$Z_R = R$
Inductors	➡	$Z_L = j(\omega \cdot L)$
Capacitors	➡	$Z_C = -j\left(\frac{1}{\omega \cdot C}\right)$

Note that the impedance of a resistor is purely real, while the impedance of either an inductor or a capacitor is purely imaginary.

Phasor Analysis with Complex Impedances

When performing a **phasor analysis**, multiple load elements are often combined into single equivalent impedances that have both resistive and reactive components, such that:



$$Z = R + jX$$

Complex number expressed in rectangular form

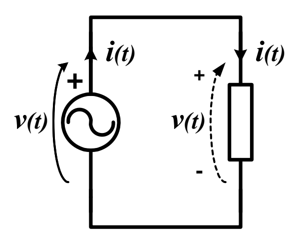
where: ***R*** is the **resistive component** of the load, and
X is the **reactive component** of the load.

Note that, when expressed in polar form, the angle of the impedance is the difference angle $\boldsymbol{\theta}$.

$$Z = \frac{\tilde{V}}{\tilde{I}} = \frac{V\angle\phi}{I\angle\delta} = \frac{V}{I}\angle(\phi-\delta) = \frac{V}{I}\angle\theta = |Z|\angle\theta$$

Complex Power

The term **<u>Complex Power</u>** is used to characterize both the real power and the reactive power produced or consumed by a single element in an AC circuit.



$$\tilde{V} = V\angle\phi$$

$$\tilde{I} = I\angle\delta$$

$$\theta = \phi - \delta$$

$$Z = R + jX$$

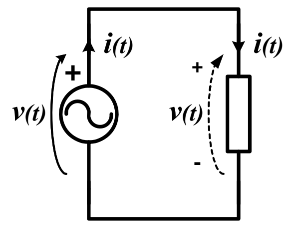
Complex Power, ***S***, is typically expressed as a complex number in rectangular form:

$$S = P + jQ$$

where: ***P*** is **real power**, and
Q is **reactive power**.

Phasor Analysis and Complex Power

The **Complex Power** produced or consumed by a single element in an AC circuit can be defined in terms of that element's phasor voltage and current:



$$S = P + jQ = \boxed{\tilde{V} \cdot \tilde{I}^*}$$
$$= (V\angle\phi)\cdot(I\angle -\delta)$$
$$= V \cdot I \angle(\phi - \delta) = V \cdot I \angle \theta$$
$$= V \cdot I \cdot \cos\theta + j\, V \cdot I \cdot \sin\theta$$

where $\tilde{I}^*$ is the **complex conjugate** of the current $\tilde{I}$:

$$\tilde{I}^* = (I\angle\delta)^* = (I\angle -\delta)$$

$\tilde{V} = V\angle\phi$

$\tilde{I} = I\angle\delta$

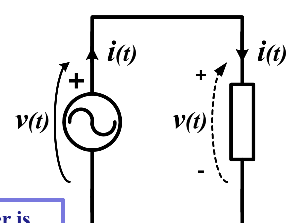
$\theta = \phi - \delta$

$Z = R + jX$

$S = P + jQ$

Apparent Power and Power Factor

Two other quantities related to complex power are often utilized when characterizing AC systems:



Apparent Power, |*S*|, is defined as the magnitude of complex power:

$$|S| = V \cdot I = \sqrt{P^2 + Q^2}$$

Apparent Power is often used when rating an AC device:
$|S|_{rated} = V_{rated} \cdot I_{rated}$

Power Factor, ***pf***, is defined as the ratio of an elements real power over its apparent power:

$$pf = \frac{P}{|S|} = \frac{V \cdot I \cdot \cos\theta}{V \cdot I} = \boxed{\cos\theta}$$

$\tilde{V} = V\angle\phi$

$\tilde{I} = I\angle\delta$

$\theta = \phi - \delta$

$Z = R + jX$

$S = P + jQ$

$S = \tilde{V} \cdot \tilde{I}^*$

Leading or Lagging Power Factor

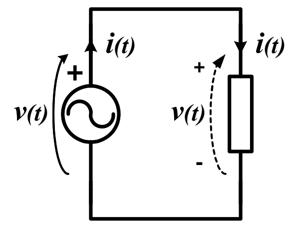
Power Factor is often characterized by a qualifier, either **leading** or **lagging**.

A **leading** power factor exists for a capacitive load where the current waveform is "leading" the voltage, resulting in a negative difference angle $\boldsymbol{\theta}$:

$$-90° \leq \theta < 0°$$

A **lagging** power factor exists for an inductive load where the current waveform is "lagging" the voltage, resulting in a positive difference angle $\boldsymbol{\theta}$:

$$0° < \theta \leq +90°$$



$$\tilde{V} = V\angle\phi$$

$$\tilde{I} = I\angle\delta$$

$$\theta = \phi - \delta$$

For a purely resistive load, the difference angle $\theta = 0°$ resulting in a "unity" power factor $pf = \cos\theta = \cos 0° = 1$

Summary of Complex Power Equations

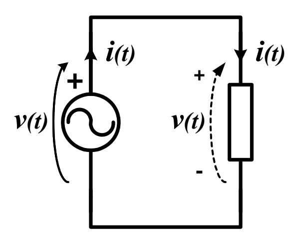
Complex Power (S): $S = P + jQ = \tilde{V} \cdot \tilde{I}^*$

Real Power (P): $P = V \cdot I \cdot \cos\theta$

Reactive Power (Q): $Q = V \cdot I \cdot \sin\theta$

Apparent Power ($|S|$): $|S| = V \cdot I = \sqrt{P^2 + Q^2}$

Power Factor (pf): $pf = \cos\theta$



$$\tilde{V} = V\angle\phi$$

$$\tilde{I} = I\angle\delta$$

$$\theta = \phi - \delta$$

$$Z = R + jX$$