ECET 3410

High Frequency Systems

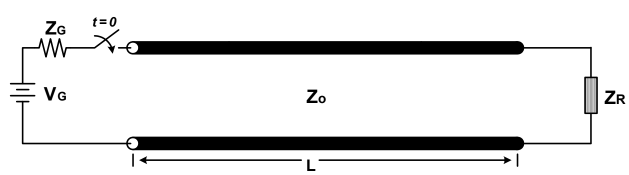
AC Sourced

Transmission Lines

1

Review

The concept of a **transmission line** was introduced during the previous presentation, with a focus on both the transient and steady-state operation of a **lossless** transmission line that was being supplied by a **DC source** that was initially energized at some arbitrary point in time.



2

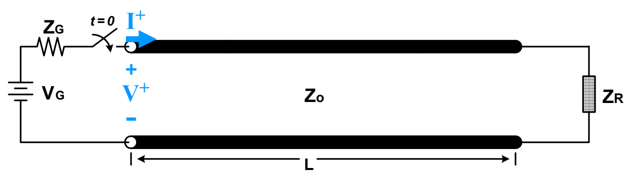
Review

Based on that discussion:

- When a DC source is initially connected to the sending-end of a line, an **initial voltage** potential, $\boldsymbol{V^{+}}$, will appear across the sending-end, and in order to build-up the charge-difference required for that voltage potential to exist, an **initial current**, $\boldsymbol{I^{+}}$, will begin to flow into the line, the magnitudes of which can be determined based on both the **source's electrical parameters** and the **characteristic impedance**, $\boldsymbol{Z_o}$, of the transmission line.

$$V^{+} = V_G \cdot \frac{Z_o}{Z_o + Z_G}$$

$$\frac{V^{+}}{I^{+}} = Z_o$$



3

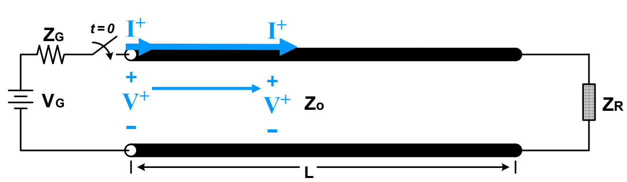
Review

Based on that discussion:

- And, as current continues to flow into the sending-end, the charge that previously entered will be pushed further down the line, resulting in what appears to be **current flowing** further and further **down the line**. And since the portion of the line across which a charge-difference exists will increase as the current pushes further down the line, a **voltage potential** will also appear to simultaneously **move down the line** with the flow of current.

$$V^{+} = V_G \cdot \frac{Z_o}{Z_o + Z_G}$$

$$\frac{V^{+}}{I^{+}} = Z_o$$



4

Review

Based on that discussion:

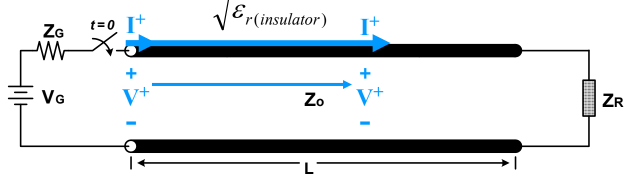
- Thus, the concept of **traveling waveforms** was first presented in the form of an **incident voltage**, V^+, and an **incident current**, I^+.

Note that, for a lossless transmission line, the rate at which the voltage and current waveforms appear to travel down the line (i.e. – **velocity**) was a function of the material properties of the insulating material that surrounded the conductors.

$$v = \frac{c}{\sqrt{\varepsilon_{r(insulator)}}}$$

$$V^+ = V_G \cdot \frac{Z_o}{Z_o + Z_G}$$

$$\frac{V^+}{I^+} = Z_o$$



5

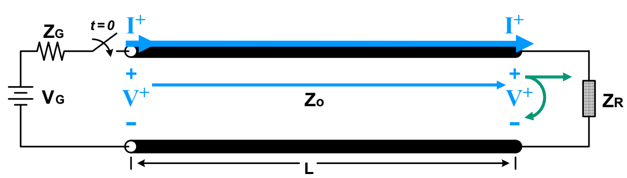
Review

Based on that discussion:

- When the incident voltage and current **reach the receiving-end** of the line, two things can happen depending on whether or not the load impedance is matched to the characteristic impedance:
  - The **energy** associated with the traveling waves **will be delivered to the load**, or
  - The **energy** associated with the traveling waves **will reflect off of the load** and travel back towards the sending-end.

$$V^+ = V_G \cdot \frac{Z_o}{Z_o + Z_G}$$

$$\frac{V^+}{I^+} = Z_o$$



6

Review

Based on that discussion:

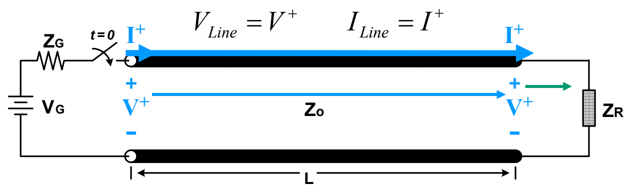
- If the load is **matched** to the characteristic impedance of the line:

$$Z_R = Z_o$$

then **all** of the **energy** associated with the traveling waves will be **delivered to the load**, and **steady-state operation** will be achieved such that the entire line will be charged-up to the value of the incident voltage, and the incident current will be flowing through the entire line.

$$V^+ = V_G \cdot \frac{Z_o}{Z_o + Z_G}$$

$$\frac{V^+}{I^+} = Z_o$$



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Review

Based on that discussion:

- But, if the load is **not matched** to the impedance of the line:

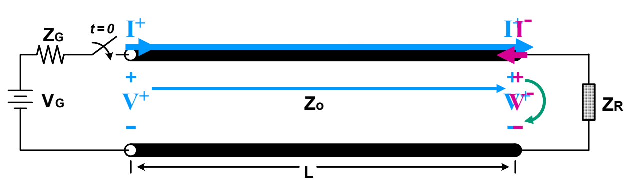
$$Z_R \neq Z_o$$

then a **portion** of the **traveling waves' energy will reflect** off of the load **and travel back towards the sending-end**.

Note that the reflected energy is characterized in terms of a **reflected voltage, *V*⁻**, and a **reflected current, *I*⁻**.

$$V^+ = V_G \cdot \frac{Z_o}{Z_o + Z_G}$$

$$\frac{V^+}{I^+} = Z_o$$



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Review

Based on that discussion:

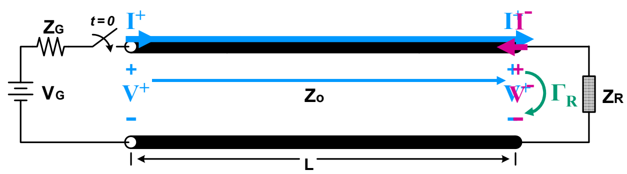
- The amount of the wave that reflects off of the load can be determined based of a **reflection coefficient, Γ_R**:

$$\Gamma_R = \frac{V^-}{V^+} = \frac{I^-}{I^+} = \frac{Z_R - Z_o}{Z_R + Z_o}$$

such that: $V^- = V^+ \cdot \Gamma_R$ and $I^- = I^+ \cdot \Gamma_R$

$$V^+ = V_G \cdot \frac{Z_o}{Z_o + Z_G}$$

$$\frac{V^+}{I^+} = Z_o$$



$$\Gamma_R = \frac{Z_R - Z_o}{Z_R + Z_o}$$

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Review

Based on that discussion:

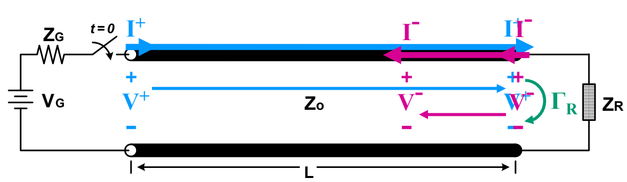
- And once the **reflected waveforms** are created, they will **travel back towards the sending-end** of the line.

Note that, as the reflected waves travel from the receiving-end, the **voltage and current on the line** will change since:

$V_X = V_X{}^+ + V_X{}^-$ and $I_X = I_X{}^+ - I_X{}^-$

$$V^+ = V_G \cdot \frac{Z_o}{Z_o + Z_G}$$

$$\frac{V^+}{I^+} = Z_o$$



$$V^- = V^+ \cdot \Gamma_R$$

$$\Gamma_R = \frac{Z_R - Z_o}{Z_R + Z_o}$$

$$I^- = I^+ \cdot \Gamma_R$$

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Review

Based on that discussion:

- Eventually the **reflected waveforms** will reach the sending-end, and if the source is **matched** to the impedance of the line:

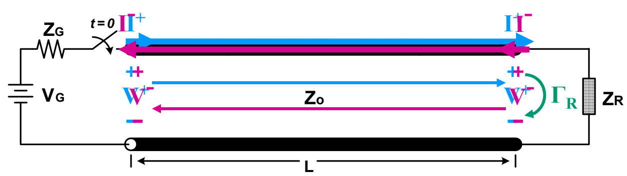
$$Z_G = Z_o$$

then **all** of the **energy** of the reflected waves will be **delivered back to the source**, and **steady-state operation** will be achieved such that:

$$V_{Line} = V^+ + V^- \quad \text{and} \quad I_{Line} = I^+ - I^-$$

$$V^+ = V_G \cdot \frac{Z_o}{Z_o + Z_G}$$

$$\frac{V^+}{I^+} = Z_o$$



$$V^- = V^+ \cdot \Gamma_R$$

$$\Gamma_R = \frac{Z_R - Z_o}{Z_R + Z_o}$$

$$I^- = I^+ \cdot \Gamma_R$$

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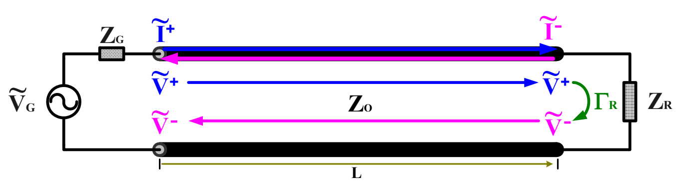
AC Sourced Transmission Lines

It turns out that **all of the theory** that was presented for a DC supplied transmission line will also apply to a transmission line that is supplied by an AC source.

But, there is one important consideration that must be taken into account when analyzing AC-sourced lines:

since the source is sinusoidally varying, both the **incident wave** and the **reflected wave** will also be **sinusoidally varying**.



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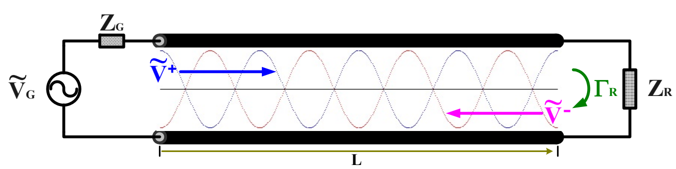
AC Sourced Transmission Lines

Thus, unlike DC-sourced lines that experience constant voltages and currents on the entire line under steady-state conditions, the **voltages** and **currents** on the **AC-sourced lines** will be **time varying**, even **under steady-state conditions**.

And, it is the time-varying nature of these voltages and currents that will cause the **AC-sourced lines** to **no longer react like a pair of ideal wires under steady-state conditions**, even if considered lossless.



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AC Transmission Lines
Part I

Transmission Line Modeling

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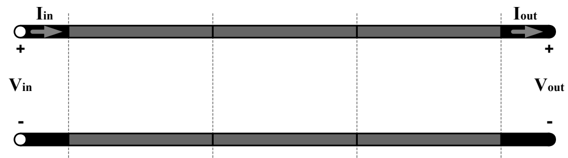
Transmission Line Modeling



Due to the sinusoidally-varying nature of an AC source and the resulting complex mathematics, we cannot approach to concept of AC-supplied transmission lines in the same simple manner that we approached DC-supplied lines.

Instead, we will apply **finite-element modeling theory** to try to predict the manner in which a transmission line will react to an AC source.

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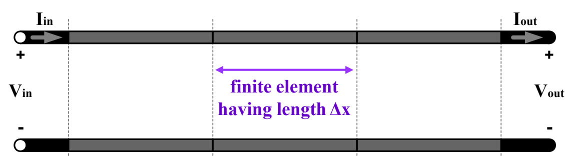
Transmission Line Modeling



In order to accurately **model** the effects of a transmission line on a wave that is propagating down the line, the line is typically broken down in the **small incremental sections** (finite elements) that are connected together (in-series) to form the overall line.

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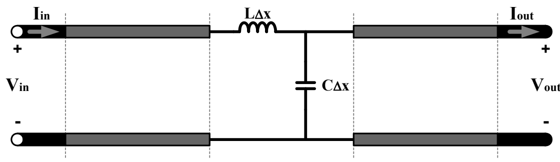
Incremental Transmission Line Model



The two primary characteristics of a line that must be included in the model in order to account for the finite propagation velocity of a wave that propagates down the line are the **inductance** and the **capacitance** of the line, where:

L is the inductance per unit length (H/length),
C is the capacitance per unit length (F/length), and
Δx is the length of the incremental section.

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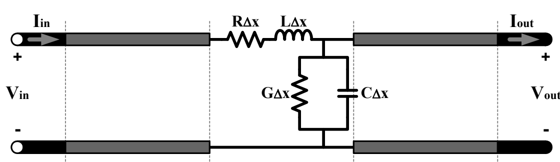
Incremental Transmission Line Model



Additionally, for a “lossy” line, the **resistance** of the conductors and the **conductance** of the insulation are also incorporated into the model, such that:

R is the resistance per unit length (Ω/length),
G is the conductance per unit length (S/length),

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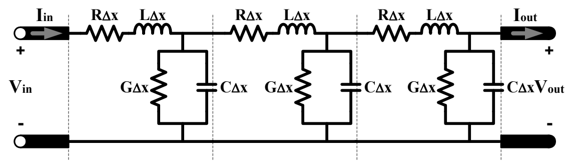
Incremental Transmission Line Model



The overall operation of the line can then be accurately predicted by replacing each incremental section by the specified model provided that the **length of each section is small compared to the wavelength** of the applied waveform.

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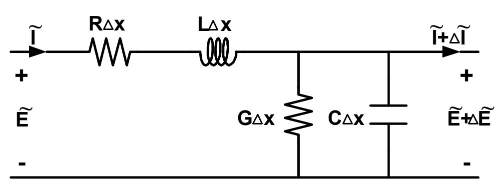
Steady-State AC Model
of a Uniform Transmission Line



R ≡ Ω/Δx
L ≡ H/Δx
G ≡ S/Δx
C ≡ F/Δx

The above figure shows the model for an incremental section of transmission line with the voltages and currents defined at both the sending-end and the receiving-end terminals, such that $\tilde{E}$ and $\tilde{I}$ are the **phasor values** of the **voltage** and **current** seen the sending-end of the line, and $\Delta\tilde{E}$ and $\Delta\tilde{I}$ are the **change in the voltage and current** from the sending to the receiving end.

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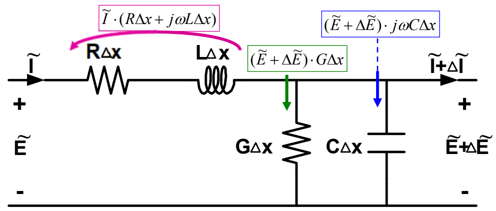
Steady-State AC Model
of a Uniform Transmission Line



R ≡ Ω/Δx
L ≡ H/Δx
G ≡ S/Δx
C ≡ F/Δx

The voltage and current at the receiving-end of each line section can be **expressed in terms of the sending-end voltage and current** and the **line parameters** as follows:

$$\tilde{E} + \Delta\tilde{E} = \tilde{E} - \tilde{I}\cdot(R\Delta x + j\omega L\Delta x)$$

$$\tilde{I} + \Delta\tilde{I} = \tilde{I} - (\tilde{E} + \Delta\tilde{E})\cdot(G\Delta x + j\omega C\Delta x)$$

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Steady-State AC Model Solution

Given the following **equations** derived for the incremental model of a **uniform transmission line**:

$$\tilde{E} + \Delta\tilde{E} = \tilde{E} - \tilde{I}\cdot(R\Delta x + j\omega L\Delta x)$$

$$\tilde{I} + \Delta\tilde{I} = \tilde{I} - (\tilde{E} + \Delta\tilde{E})\cdot(G\Delta x + j\omega C\Delta x)$$

If the **second order terms** are assumed to be small ($\Delta\tilde{E}\Delta x \approx 0$) and thus **ignored**, then:

$$\tilde{E} + \Delta\tilde{E} = \tilde{E} - \tilde{I}\cdot(R\Delta x + j\omega L\Delta x)$$

$$\tilde{I} + \Delta\tilde{I} = \tilde{I} - \tilde{E}\cdot(G\Delta x + j\omega C\Delta x)$$

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Steady-State AC Model Solution

The resultant equations:

$$\tilde{E} + \Delta\tilde{E} = \tilde{E} - \tilde{I} \cdot (R\Delta x + j\omega L\Delta x)$$

$$\tilde{I} + \Delta\tilde{I} = \tilde{I} - \tilde{E} \cdot (G\Delta x + j\omega C\Delta x)$$

are often **simplified** by substituting:

$$Z = R + j\omega L \qquad Y = G + j\omega C$$

series impedance parallel admittance

resulting in:

$$\tilde{E} + \Delta\tilde{E} = \tilde{E} - \tilde{I}Z\Delta x$$

$$\tilde{I} + \Delta\tilde{I} = \tilde{I} - \tilde{E}Y\Delta x$$

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Steady-State AC Model Solution

Given the equations:

$$\tilde{E} + \Delta\tilde{E} = \tilde{E} - \tilde{I}Z\Delta x \qquad \tilde{I} + \Delta\tilde{I} = \tilde{I} - \tilde{E}Y\Delta x$$

by canceling like terms, we may solve for the **change in the voltage and current** from sending-end to receiving-end of an incremental section of line ($\Delta\tilde{E}$ and $\Delta\tilde{I}$ as a function of Δx):

$$\Delta\tilde{E} = -\tilde{I}Z\Delta x \qquad \Delta\tilde{I} = -\tilde{E}Y\Delta x$$

from which we can define **rates-of-change per unit length**:

$$\frac{\Delta\tilde{E}}{\Delta x} = -Z \cdot \tilde{I} \qquad \frac{\Delta\tilde{I}}{\Delta x} = -Y \cdot \tilde{E}$$

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Steady-State AC Model Solution

Given the equations:

$$\frac{\Delta \tilde{E}}{\Delta x} = -Z \cdot \tilde{I} \qquad \frac{\Delta \tilde{I}}{\Delta x} = -Y \cdot \tilde{E}$$

by allowing the length of the incremental section to become infinitely small ($\Delta x \to 0$), we can define the following **1st order differential equations** relating to the rates of change in the voltage and current as a function of position on a uniform line:

$$\lim_{\Delta x \to 0} \frac{\Delta \tilde{E}}{\Delta x} = \frac{d\tilde{E}}{dx} = -Z \cdot \tilde{I} \qquad \lim_{\Delta x \to 0} \frac{\Delta \tilde{I}}{\Delta x} = \frac{d\tilde{I}}{dx} = -Y \cdot \tilde{E}$$

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Steady-State AC Model Solution

The 1st order differential equations:

$$\frac{d\tilde{E}}{dx} = -Z \cdot \tilde{I} \qquad \frac{d\tilde{I}}{dx} = -Y \cdot \tilde{E}$$

can be combined into a single **2nd order differential equation** by taking the derivative of both sides of the first equation, solving for $d\tilde{I}/dx$, and substituting the result into the second equation:

$$\frac{d^2 \tilde{E}}{dx^2} = Z \cdot Y \cdot \tilde{E}$$

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Steady-State AC Model Solution

The **2nd order differential equation**:

$$\frac{d^2 \widetilde{E}}{dx^2} = Z \cdot Y \cdot \widetilde{E}$$

has the following **general solution**:

$$\widetilde{E}(x) = A_1 \cdot e^{-\sqrt{Z \cdot Y} \cdot x} + A_2 \cdot e^{+\sqrt{Z \cdot Y} \cdot x}$$

which can be utilized to define the voltage on a uniform transmission line as a function of position on the line.

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Steady-State AC Model Solution

Note that the equation:

$$\widetilde{E}(x) = A_1 \cdot e^{-\sqrt{Z \cdot Y} \cdot x} + A_2 \cdot e^{+\sqrt{Z \cdot Y} \cdot x}$$

has **two terms**, similar to the equation:

$$\widetilde{E}(x) = \widetilde{E}_x^+ + \widetilde{E}_x^-$$

which defines the **steady-state voltage** at position "x" on a transmission-line as the **sum** of an **incident voltage** and a **reflected voltage**.

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Steady-State AC Model Solution

By substituting the equation:

$$\widetilde{E}(x) = A_1 \cdot e^{-\sqrt{Z \cdot Y} \cdot x} + A_2 \cdot e^{+\sqrt{Z \cdot Y} \cdot x}$$

into the 1st order differential equation:

$$\frac{d\widetilde{E}}{dx} = -Z \cdot \widetilde{I}$$

and solving for **current**, an equation can also be defined for the current flowing in a uniform transmission line as a function of position on the line:

$$\widetilde{I}(x) = -\frac{1}{Z} \cdot \frac{d\widetilde{E}}{dx} = \frac{A_1 \cdot e^{-\sqrt{Z \cdot Y} \cdot x}}{\sqrt{Z/Y}} - \frac{A_2 \cdot e^{+\sqrt{Z \cdot Y} \cdot x}}{\sqrt{Z/Y}}$$

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Steady-State AC Model Solution

Note that the equation:

$$\widetilde{I}(x) = \frac{A_1 \cdot e^{-\sqrt{Z \cdot Y} \cdot x}}{\sqrt{Z/Y}} - \frac{A_2 \cdot e^{+\sqrt{Z \cdot Y} \cdot x}}{\sqrt{Z/Y}}$$

also has two terms, similar to the equation:

$$\widetilde{I}(x) = \widetilde{I}_x^+ - \widetilde{I}_x^-$$

which defines the **steady-state current** at position "x" on a transmission line as the **difference** between an **incident current** and a **reflected current**.

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Steady-State AC Model Solution

Furthermore, note that the **first term** of the **current equation**:

$$\tilde{I}(x) = \frac{A_1 \cdot e^{-\sqrt{Z \cdot Y} \cdot x}}{\sqrt{Z/Y}} - \frac{A_2 \cdot e^{+\sqrt{Z \cdot Y} \cdot x}}{\sqrt{Z/Y}}$$

is **equal** to the **first term** of the **voltage equation**:

$$\tilde{E}(x) = A_1 \cdot e^{-\sqrt{Z \cdot Y} \cdot x} + A_2 \cdot e^{+\sqrt{Z \cdot Y} \cdot x}$$

divided by the **constant** $\sqrt{Z/Y}$.

The same relationship **also** holds **true** for the **second terms** of the respective equations.

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Steady-State AC Model Solution

Thus, given the equations:

$$\tilde{E}(x) = A_1 \cdot e^{-\sqrt{Z \cdot Y} \cdot x} + A_2 \cdot e^{+\sqrt{Z \cdot Y} \cdot x} \qquad \tilde{I}(x) = \frac{A_1 \cdot e^{-\sqrt{Z \cdot Y} \cdot x}}{\sqrt{Z/Y}} - \frac{A_2 \cdot e^{+\sqrt{Z \cdot Y} \cdot x}}{\sqrt{Z/Y}}$$

and the relationships:

$$\tilde{I}_x^+ = \frac{\tilde{E}_x^+}{Z_o} \qquad \tilde{I}_x^- = \frac{\tilde{E}_x^-}{Z_o}$$

it can be seen that both of the **first terms** of the derived equations relate to **incident voltage and current waveforms** while the **second terms** relate to **reflected voltage and current waveforms**.

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Steady-State AC Solution

Relationships

And, based on those results, the following terms may be defined:

Characteristic Impedance:

$$Z_o = \sqrt{Z/Y}$$

Propagation Constant (γ):

$$\gamma = \sqrt{Z \cdot Y} = \alpha + j\beta$$

where: α is the **attenuation constant** of the line in **nepers/meter**, and

β is the **phase constant** for the line in **radians/meter**.

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Steady-State AC Model Solution

If the new expressions for characteristic impedance and propagation constant are substituted into the equations for voltage and current, then the **general solutions** for **voltage and current** on a transmission line, as a function of position, ***x***, are:

$$\tilde{E}(x) = \tilde{E}_x^+ + \tilde{E}_x^- = A_1 \cdot e^{-\gamma \cdot x} + A_2 \cdot e^{+\gamma \cdot x}$$

$$\tilde{I}(x) = \tilde{I}_x^+ - \tilde{I}_x^- = \frac{A_1 \cdot e^{-\gamma \cdot x}}{Z_o} - \frac{A_2 \cdot e^{+\gamma \cdot x}}{Z_o}$$

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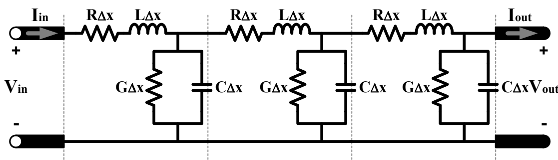
Steady-State AC Model Solution



Thus, the governing relationships for **AC voltage** and **current** on a transmission-line as a function of position are:

$$\tilde{E}(x) = \tilde{E}_x^+ + \tilde{E}_x^- = A_1 \cdot e^{-\gamma \cdot x} + A_2 \cdot e^{+\gamma \cdot x} \qquad \tilde{I}(x) = \tilde{I}_x^+ - \tilde{I}_x^- = \frac{A_1 \cdot e^{-\gamma \cdot x}}{Z_o} - \frac{A_2 \cdot e^{+\gamma \cdot x}}{Z_o}$$

$$\tilde{I}_x^+ = \frac{\tilde{E}_x^+}{Z_o} \qquad \tilde{I}_x^- = \frac{\tilde{E}_x^-}{Z_o} \qquad Z_o = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \qquad \gamma = \sqrt{Z \cdot Y} = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta$$

Note that, if the characteristic impedance is assumed to be purely real, then the **power** associated with the incident and reflected waveforms will be equal to:

Incident Power $P_x^+ = E_x^+ I_x^+ = \frac{(E_x^+)^2}{Z_o}$ **Reflected Power** $P_x^- = E_x^- I_x^- = \frac{(E_x^-)^2}{Z_o}$

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AC Transmission Lines
Part II

Traveling AC Waves
on a
Matched Transmission Line

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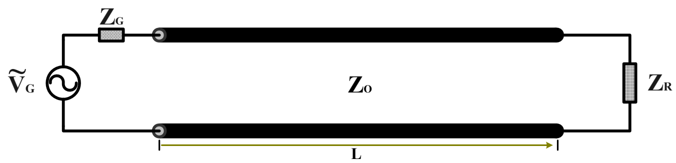
Analysis of an AC-sourced Transmission-Line



During the previous lecture, we began to analyze the steady-state operation of a transmission-line that was supplied by a source that was sinusoidally-varying.

Note that $\tilde{V}_G$ is the **phasor** representation of the sinusoidally-varying source voltage:

$$v_G(t) = \sqrt{2} \cdot V_G \cdot \sin(\omega \cdot t + \phi_G) \quad \text{volts.}$$

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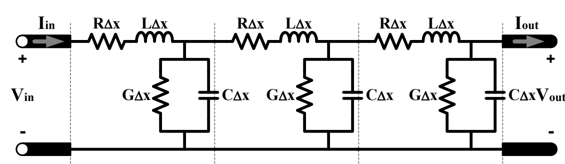
General Solutions for an AC-sourced Line



And, to facilitate that analysis, we utilized a finite element model of a transmission-line to obtain the following **general solutions** for **voltage** and **current** on that line **as a function of position**:

$$\tilde{E}(x) = A_1 \cdot e^{-\gamma \cdot x} + A_2 \cdot e^{+\gamma \cdot x} \qquad \tilde{I}(x) = \frac{A_1 \cdot e^{-\gamma \cdot x}}{Z_o} - \frac{A_2 \cdot e^{+\gamma \cdot x}}{Z_o}$$

where: $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ were constants,
γ was defined to be the **propagation constant**, and
$\boldsymbol{Z_o}$ was the **characteristic impedance** of the line.

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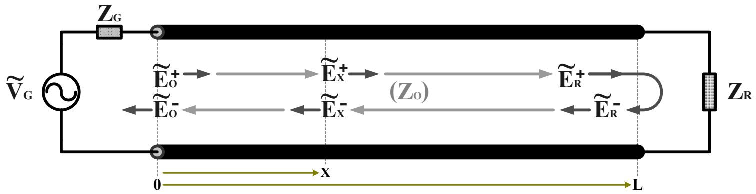
Traveling Waveforms on a Transmission-Line



Furthermore, since **actual voltage** at any position on a transmission-line is the **sum** of any **incident and reflected voltages** seen at that position on the line, it was determined that:

$$\tilde{E}(x) = \tilde{E}_x^+ + \tilde{E}_x^- = A_1 \cdot e^{-\gamma \cdot x} + A_2 \cdot e^{+\gamma \cdot x}$$

where: $\tilde{E}_x^+ = A_1 \cdot e^{-\gamma \cdot x}$ defines the **incident voltage** on the line, and

$\tilde{E}_x^- = A_2 \cdot e^{+\gamma \cdot x}$ defines the **reflected voltage** on the line.

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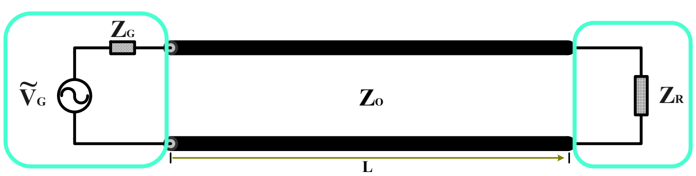
Boundary Conditions and the General Solution



$$\tilde{E}(x) = \tilde{E}_x^+ + \tilde{E}_x^- = A_1 \cdot e^{-\gamma \cdot x} + A_2 \cdot e^{+\gamma \cdot x}$$

$$\gamma = \sqrt{Z \cdot Y} = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta$$

$$Z_o = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Although both the **propagation constant**, γ, and the **characteristic impedance**, Z_o, are defined in terms of the parameters that were utilized within the model to represent the various losses associated with the transmission line, the constants A_1 and A_2 within the expression:

$$\tilde{E}(x) = \tilde{E}_x^+ + \tilde{E}_x^- = A_1 \cdot e^{-\gamma \cdot x} + A_2 \cdot e^{+\gamma \cdot x}$$

are a function of the **boundary conditions** of the system.

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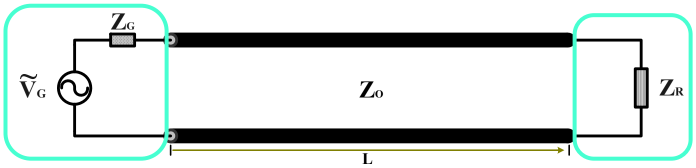
Boundary Conditions and the General Solution



$$\tilde{E}(x) = \tilde{E}_x^+ + \tilde{E}_x^- = A_1 \cdot e^{-\gamma \cdot x} + A_2 \cdot e^{+\gamma \cdot x}$$

$$\gamma = \sqrt{Z \cdot Y} = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta$$

$$Z_o = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Thus, in addition to being functions of the **parameters of the line**, the **constants A_1** and **A_2** are also functions of the **parameters of** both **the source** and **the load** that are connected to the line.

Which means, if we are given a system for which the source and load parameters are defined, then we can utilize that information in order to determine those constant values and, in-turn, the **specific solution** for voltage as a function of position on that line.

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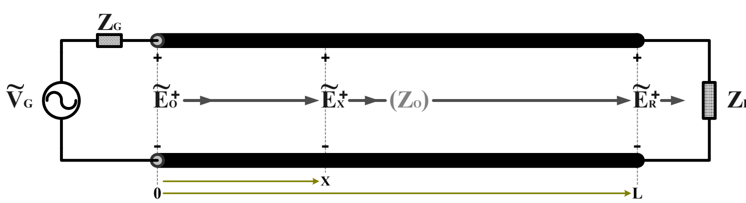
Traveling Waveforms on a Matched Line



$$Z_R = Z_o$$

Assuming that the **load impedance** is **matched** to the characteristic impedance of the line (**$Z_R = Z_o$**), then **only an incident waveform will exist** because no reflection will occur ($\tilde{E}_R^- = 0$) when the incident wave reaches the receiving-end of the line.

Thus, the **voltage** on a matched line can be expressed as:

$$\tilde{E}(x) = \tilde{E}_x^+ = A_1 \cdot e^{-\gamma \cdot x}$$

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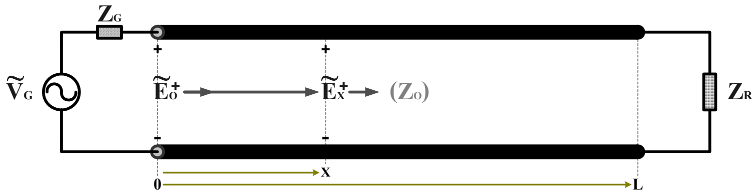
Solving for the Constant A_1



$Z_R = Z_o$

Given the expression:

$$\tilde{E}(x) = \tilde{E}_x^+ = A_1 \cdot e^{-\gamma \cdot x}$$

the constant A_1 can be determined by setting the expression equal to a **known voltage** at a specific position on the line.

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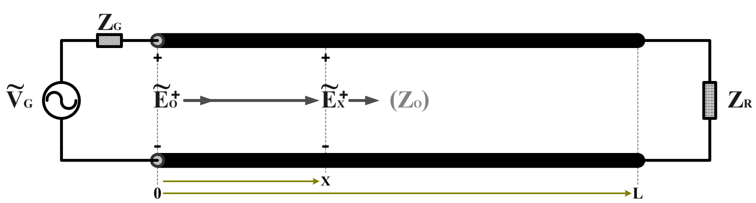
Solving for the Constant A_1



$Z_R = Z_o$

The phasor value of the **incident waveform** applied by the source, **at the sending-end of the line**, can be determined from the voltage-divider equation:

$$\tilde{E}(0) = \tilde{E}_0^+ = \tilde{E}_S^+ = \tilde{V}_G \cdot \left(\frac{Z_o}{Z_G + Z_o} \right)$$

since the impedance experienced by that incident waveform is equal to the characteristic impedance of the line.

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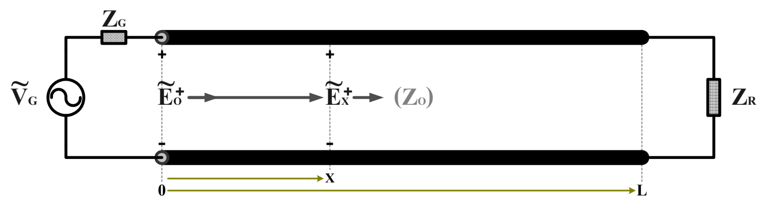
Solving for the Constant A_1



$Z_R = Z_o$

And if the incident voltage, $\widetilde{E}_o^+$, at the sending-end (x=0) of the line is known, then the **constant A_1** can be solved as follows:

$$\widetilde{E}(x) = A_1 \cdot e^{-\gamma \cdot x}$$

$$\widetilde{E}(0) = \widetilde{E}_o^+ = A_1 \cdot e^{-\gamma \cdot (0)} = A_1 \cdot 1$$

$$\therefore \quad \boxed{A_1 = \widetilde{E}_o^+}$$

Note that, although a matched load was assumed in order to simplify the problem, the method used to obtain the constant A_1 would still return the same result even if the load was mismatched ($Z_R \neq Z_o$).

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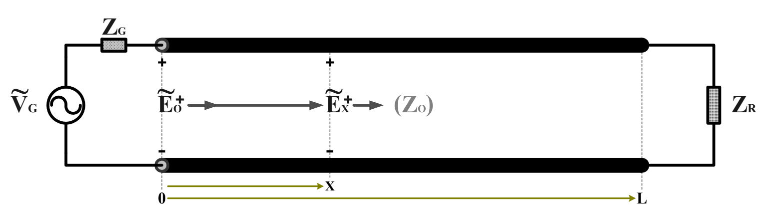
Incident Voltage on a Transmission Line



$Z_R = Z_o$

Thus, the expression for **voltage as a function of position** on a **matched transmission-line** is:

$$\widetilde{E}(x) = \widetilde{E}_o^+ \cdot e^{-\gamma \cdot x}$$

where: $\widetilde{E}_o^+$ is the **phasor value** of the **incident voltage** applied to the **sending-end of the line**, and

γ is the **propagation constant** for the line.

Although the result appears to be simple, the true nature of this propagating waveform is difficult to see unless the solution is broken down into its different components.

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Phasors and AC Voltages

Remember that a sinusoidal voltage:

$$e(t) = \sqrt{2} \cdot E \cdot \sin(\omega \cdot t + \phi)$$

can be expressed as an equivalent ***phasor voltage***:

$$\tilde{E} = E \cdot e^{j\phi} = E\angle\phi$$

where: $\tilde{E}$ is a complex number in "***polar***" form, such that
E is the **RMS magnitude** of the voltage, and
ϕ is the **phase angle** of the voltage.

Note that, although phasor values may be expressed in terms of their "peak" magnitudes, **RMS magnitudes** will be utilized in this course unless specifically stated otherwise.

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$\tilde{E}_o^+ \cdot e^{-\gamma \cdot x}$

Now, let's take a closer look at the expression $\tilde{E}(x) = \tilde{E}_o^+ \cdot e^{-\gamma \cdot x}$.

$\tilde{E}_o^+$ is the **phasor value** of the **applied incident voltage**, which can be expressed as a complex number in polar form:

$$\tilde{E}_o^+ = E_0^+\angle\phi°$$

and since $\boldsymbol{\gamma = \alpha + j\beta}$ is a complex number, we can also express the exponential term $\boldsymbol{e^{-\gamma x}}$ as a **complex number** in **polar form**:

$$e^{-\gamma \cdot x} = e^{-(\alpha + j\beta)\cdot x} = e^{-\alpha \cdot x} e^{-j\beta \cdot x} = e^{-\alpha \cdot x}\angle - \beta \cdot x$$

where: $e^{-\alpha x}$ is the **magnitude** of the **complex exponential**, and
$-\beta \cdot x$ is the **phase angle** of the **complex exponential**.

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$\tilde{E}_o^+ \cdot e^{-\gamma x}$

Thus, the **phasor voltage**, $\tilde{E}(x)$, on the matched line is simply the multiple of two complex numbers, $\tilde{E}_o^+$ and $e^{-\gamma \cdot x}$, where:

$$\tilde{E}(x) = \tilde{E}_o^+ \cdot e^{-\gamma \cdot x} = \left(E_0^+ \angle \phi°\right) \cdot \left(e^{-\alpha \cdot x} \angle - \beta \cdot x\right) = \boxed{E_0^+ \cdot e^{-\alpha \cdot x} \angle \phi° - \beta \cdot x}$$

which can be converted back into its equivalent time function:

$$\tilde{E}(x) = E_0^+ \cdot e^{-\alpha \cdot x} \angle \phi° - \beta \cdot x \quad \Leftrightarrow \quad \boxed{e(x,t) = \sqrt{2} \cdot E_0^+ \cdot e^{-\alpha \cdot x} \cdot \sin\left(\omega t + \phi° - \beta \cdot x\right)}$$

Based on this result, it can be seen that the **attenuation constant**, ***α***, affects the **magnitude** of the resultant voltage, while the **phase constant**, ***β***, affects the **phase** of the resultant voltage.

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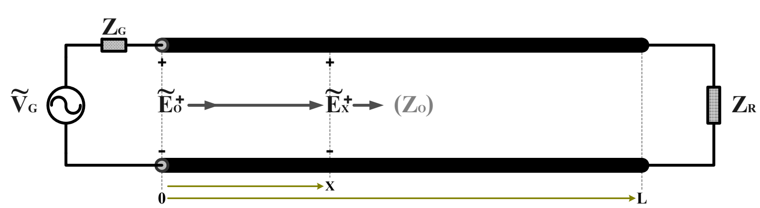
Incident Voltage on a Transmission Line



$Z_R = Z_o$

And given this sinusoidally-varying **incident voltage** that can either be expressed as a **function of time** or by its **phasor equivalent**:

$$e^+(x,t) = \sqrt{2} \cdot E_0^+ \cdot e^{-\alpha \cdot x} \cdot \sin\left(\omega t + \phi° - \beta \cdot x\right) \text{ volts, or}$$

$$\tilde{E}_x^+ = \tilde{E}_o^+ \cdot e^{-\gamma \cdot x} = E_0^+ \cdot e^{-\alpha \cdot x} \angle \phi° - \beta \cdot x \text{ volts,}$$

we can now begin to investigate the exact manner in which those constants affect the wave as it propagates down the line.

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AC Transmission Lines
Part III

Characterizing
the
Attenuation and Phase Constants

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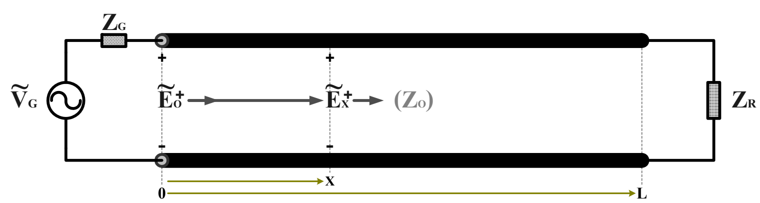
Characterizing the Propagation Constant



$Z_R = Z_o$

During the previous parts of this presentation, we created a model of an AC supplied transmission line and derived the solution for the **incident voltage** on a matched line:

[function of time] $e^+(x,t) = \sqrt{2} \cdot E_0^+ \cdot e^{-\alpha \cdot x} \cdot \sin(\omega t + \phi° - \beta \cdot x)$ volts,

[phasor equivalent] $\tilde{E}_x^+ = \tilde{E}_o^+ \cdot e^{-\gamma \cdot x} = E_0^+ \cdot e^{-\alpha \cdot x} \angle \phi° - \beta \cdot x$ volts.

We will now investigate the manner in which the **propagation constant** (***γ* = *α* + *jβ***) affects the wave as it travels down the line.

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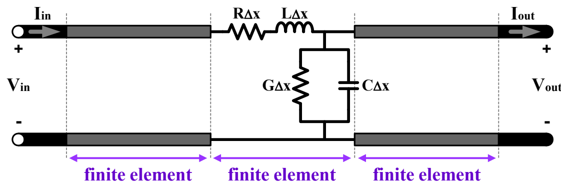
Propagation Constant



The following relationship for the **propagation constant** of a line was defined in terms of the parameters (R, L, G and C) that were included within the finite element model for a transmission line:

$$\gamma = \alpha + j\beta = \sqrt{Z \cdot Y} = \sqrt{(R + j\omega L)(G + j\omega C)}$$

where: α was the **attenuation constant** of the line in **nepers/meter**, and
β was the **phase constant** for the line in **radians/meter**.

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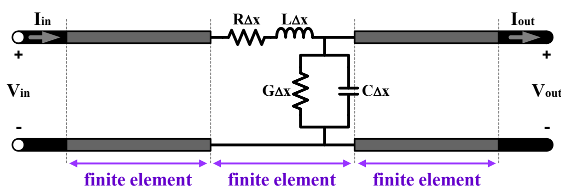
Propagation Constant Example (Lossy Line)



Given a **transmission line** with the following parameters:

***R* = 2.06 Ω/m, *L* = 365 nH/m, *G* = 0.4 μS/m, *C* = 50 pF/m**

determine the **propagation constant** for the line if the source supplying the line is operating at frequency ***f* = 400 MHz**.

$$\gamma = \sqrt{(2.06 + j(2\pi \cdot 400x10^{6})(365x10^{-9})) \cdot (0.4x10^{-6} + j(2\pi \cdot 400x10^{6})(50x10^{-12}))}$$

$$= \sqrt{(2.06 + j917.345) \cdot (0.4x10^{-6} + j0.125664)}$$

$$= 0.012072 + j10.737$$

$$\alpha = 0.012027 \ \tfrac{\text{nepers}}{\text{meter}}$$

$$\beta = 10.737 \ \tfrac{\text{radians}}{\text{meter}}$$

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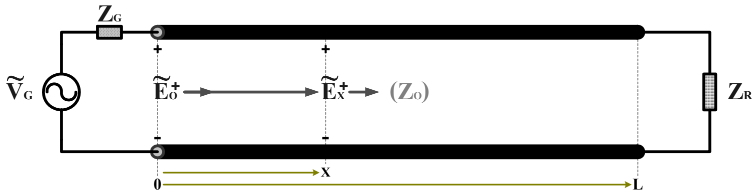
Propagation Constant



The **propagation constant**, ***γ***, of the transmission line determines the manner in which a line's physical characteristics affect any wave that is propagating (traveling) on that line.

$$\gamma = \alpha + j\beta$$

Incident Voltage applied at the Sending-End

Incident Voltage propagating on the line as a function of position → $\widetilde{E}_x^+ = \widetilde{E}_o^+ \cdot e^{-\gamma \cdot x}$

term containing the Propagation Constant that affects the incident wave as it travels distance x down the line

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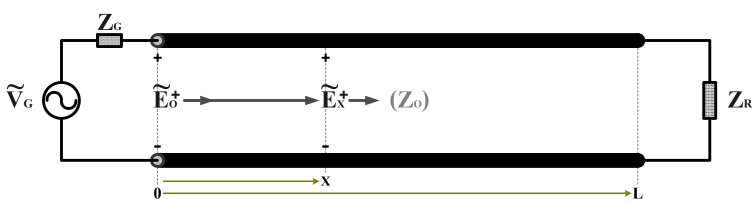
Propagation Constant



And as discussed during the previous lecture, the real component of the propagation constant, ***α***, **affects the magnitude** of the traveling wave, while the imaginary component, ***β***, **affects the phase** of the traveling wave.

$$\gamma = \alpha + j\beta$$

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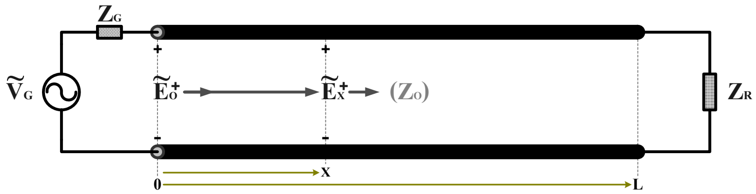
Incident Voltage on a Transmission Line



$Z_R = Z_o$

For example, if the value of the incident voltage applied by a source to the sending-end of a line is $\tilde{E}_o^+ = E_0^+ \angle \phi°$, then the **phasor value** of the **incident voltage** on the line as a **function of position** can be defined as follows:

magnitude affected by α

Incident Voltage propagating on the line as a function of position → $\tilde{E}_x^+ = \tilde{E}_o^+ \cdot e^{-\gamma \cdot x} = E_0^+ \cdot e^{-\alpha \cdot x} \angle \phi° - \beta \cdot x$ volts

phase affected by β

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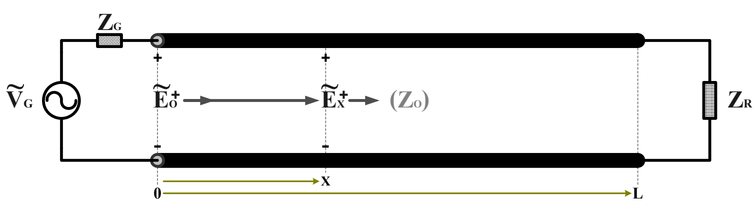
Incident Voltage on a Transmission Line



$Z_R = Z_o$

Similarly, if the phasor representation of the **incident voltage**:

$$\tilde{E}_x^+ = \tilde{E}_o^+ \cdot e^{-\gamma \cdot x} = E_0^+ \cdot e^{-\alpha \cdot x} \angle \phi° - \beta \cdot x \text{ volts,}$$

is instead expressed as a **function of time**, the effects of the attenuation and phase constants can also be seen:

magnitude affected by α

Incident Voltage propagating on the line as a function of position → $e^+(x,t) = \sqrt{2} \cdot E_0^+ \cdot e^{-\alpha \cdot x} \cdot \sin\left(\omega t + \phi° - \beta \cdot x\right)$ volts.

phase affected by β

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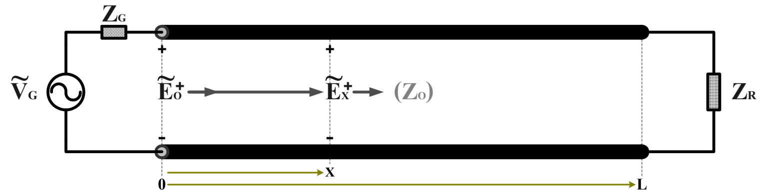
Characterizing Attenuation & Phase Constant



$Z_R = Z_o$

Since the real and imaginary components of the propagation constant affect different parts of the incident voltage expression, it is often easier to investigate their effects individually.

Thus, in order to **characterize the overall effect** of the **propagation constant** on a wave as it propagates on a line, we will begin our discussion by first considering an **incident voltage** as it propagates down a **lossless line** ($\boldsymbol{\alpha} = 0$).

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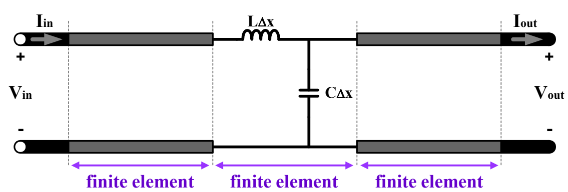
Propagation Constant on a Lossless Line



If a line is considered to be **lossless**, then both the **resistance**, ***R***, of the line's conductors and the **conductance**, ***G***, of the line's insulation will both be **zero**, resulting in:

$$\begin{aligned}\gamma &= \sqrt{(R + j\omega L)(G + j\omega C)} \\ &= \sqrt{(0 + j\omega L)(0 + j\omega C)} \\ &= \sqrt{(j\omega L)(j\omega C)} = \sqrt{j^2\omega^2 LC} \\ &= 0 + j\omega\sqrt{LC}\end{aligned}$$

Although both the resistance and the conductance terms in the model are assumed to be zero ($R = 0$, $G = 0$) for a lossless line, the **inductance**, ***L***, and the **capacitance**, ***C***, terms in the model will be **non-zero** regardless of whether the line is lossy or lossless.

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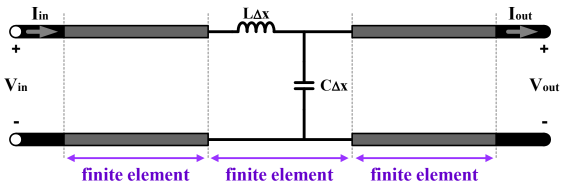
Propagation Constant on a Lossless Line



Thus, for a **lossless line**:

$$\gamma = \alpha + j\beta = 0 + j\omega\sqrt{LC}$$

such that: the **attenuation constant** $\alpha = 0$

the **phase constant** $\beta = \omega\sqrt{LC}$

For a **lossless line**, the **propagation constant** will be **purely imaginary**, resulting in an **attenuation constant** that is **zero** (**$\alpha = 0$**) and a **phase constant** (**β**) that is **proportional to frequency**.

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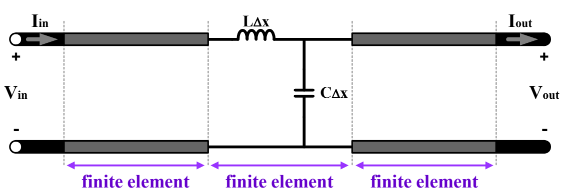
Propagation Constant Example (Lossless Line)



Given a **lossless transmission line** with the following parameters:

L **= 365 nH/m,** ***C*** **= 50 pF/m**

determine the **propagation constant** for the line is the source connected to the line is operating at frequency ***f*** **= 400 MHz.**

$$\begin{aligned}\gamma_{lossless\ line} &= 0 + j\omega\sqrt{LC}\\ &= 0 + j(2\pi \cdot 400x10^{6})\sqrt{(365x10^{-9})(50x10^{-12})}\\ &= 0 + j10.737\end{aligned}$$

$\alpha = 0 \ \frac{\text{nepers}}{\text{meter}}$

$\beta = 10.737 \ \frac{\text{radians}}{\text{meter}}$

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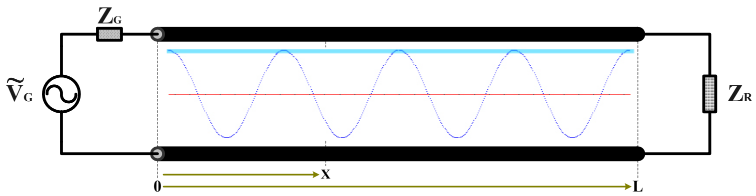
Attenuation on a Lossless Line



$Z_R = Z_o$

Since the general solution for **incident voltage** on a line is:

$$e^{+}(x,t) = \sqrt{2} \cdot E_0^{+} \cdot e^{-\alpha \cdot x} \cdot \sin(\omega t + \phi° - \beta \cdot x)$$

if the line is **lossless** (***α=0***), then the **magnitude** of the incident voltage will be **constant** (i.e. – it will not vary with position) since:

$$e^{-\alpha \cdot x} = e^{-(0) \cdot x} = 1 \quad \Rightarrow \quad e^{+}(x,t) = \sqrt{2} \cdot E_0^{+} \cdot \sin(\omega t + \phi° - \beta \cdot x)$$

magnitude does not vary with position x

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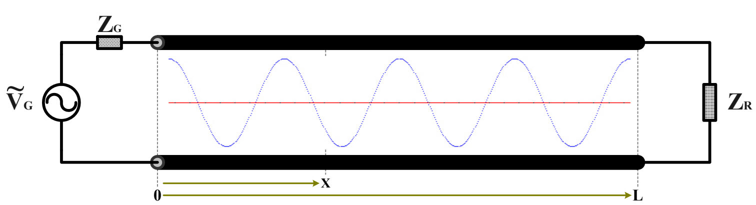
Characterizing the Phase Constant



$Z_R = Z_o$

But, given the solution for **incident voltage** on a lossless line:

$$e^{+}(x,t) = \sqrt{2} \cdot E_0^{+} \cdot \sin(\omega t + \phi° - \beta \cdot x)$$

how does the expression that contains the **phase constant**, ***β***, term within the sine function affect the wave as it propagates down the line?

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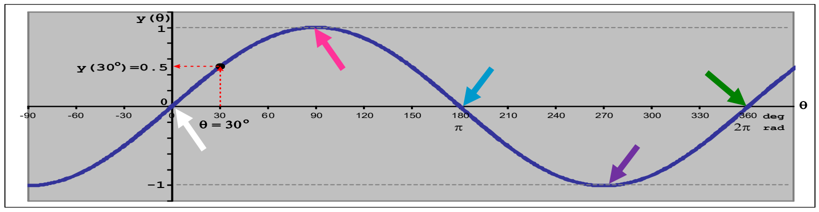
The Phase of a Sine Function



The **phase** (**θ**) of the sine function:

phase

$$y(\theta) = \sin(\theta)$$

defines the progression of a function through its periodic variation.

As **θ** varies from **0°→ 360°**, **sin(θ)** varies from **0→1→0→−1→0**, repeating again with every additional **360°** increase in **θ.**

θ may also be defined in **radians** such that **360° ≡ 2·*π* radians**.

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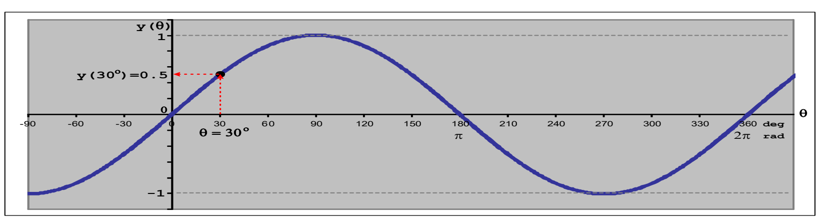
The Phase of a Sine Function



When sine is defined as a **function of time**:

$$y(t) = \sin(\omega t)$$

Since ω is defined in $^{\text{radians}}/_{\text{second}}$, ωt provides an angle in **radians**.

the **angular frequency**, ω, defines the rate at which the function progresses through its periodic variation, such that the **period**, T, or length of time required to progress through one complete cycle of the waveform is:

$$T = \frac{2\pi}{\omega} \text{ seconds.}$$

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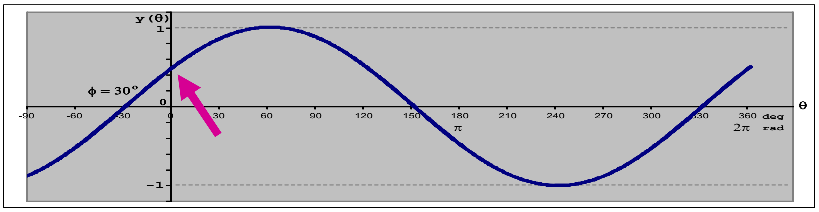
The Phase of a Sine Function



When a **phase shift**, ϕ, is added into the function:

$$y(t) = \sin(\omega t + \phi)$$

the sine function begins part way through its progression at the arbitrary reference time $\boldsymbol{t} = \mathbf{0}$.

But, why does the **incident voltage** $e^{+}(x,t) = \sqrt{2} \cdot E_0^{+} \cdot \sin(\omega t + \phi° - \beta \cdot x)$ expression have a "**$-\beta \cdot x$** term" that is a **function of position**?

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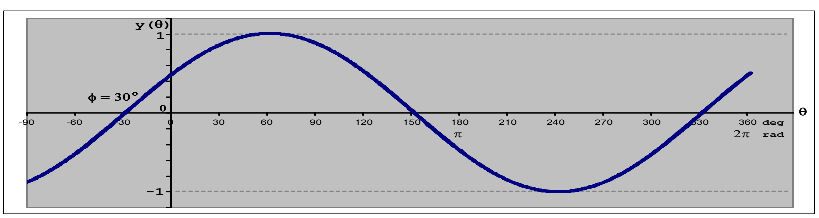
Incident Voltage at the Sending-End



If, as previously derived, the phasor value of the **incident voltage** applied by the source to the **sending-end of the line** (**x = 0**) is:

$$\tilde{E}(0) = \tilde{E}_0^{+} = \tilde{V}_G \cdot \left(\frac{Z_o}{Z_G + Z_o} \right) \quad \Rightarrow \quad \tilde{E}_o^{+} = E_0^{+} \angle \phi° \text{ volts,}$$

then the time function that describes the **incident voltage** at the **sending-end** of the line is:

$$e_o^{+}(t) = \sqrt{2} \cdot E_0^{+} \cdot \sin(\omega t + \phi°) \text{ volts.}$$

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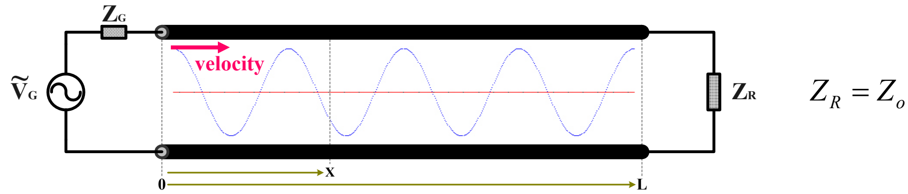
Incident Voltage on a Transmission Line



But, as the source is applying the **incident voltage**:

$$e_o^{+}(t) = \sqrt{2} \cdot E_0^{+} \cdot \sin(\omega t + \phi°) \text{ volts.}$$

to the sending-end of the line, the instantaneous voltage (and the associated current waveform) seen at the sending-end begins to propagate down the line at a **finite velocity**.

And since the voltage varies sinusoidally, it creates an **incident voltage** on the line that **varies sinusoidally as a function of position**.

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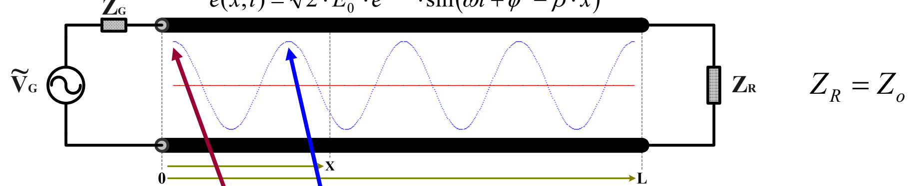
Incident Voltage on a Transmission Line

$$e(x,t) = \sqrt{2} \cdot E_0^{+} \cdot e^{-\alpha \cdot x} \cdot \sin(\omega t + \phi° - \beta \cdot x)$$



If the incident voltage on the line is plotted as a function of position at some arbitrary time t_o… **(as shown in the above figure)**

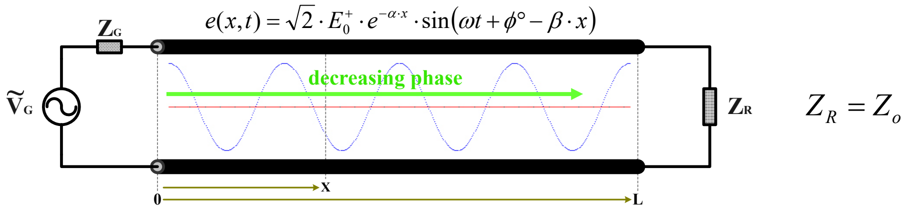
then the instantaneous voltage shown at this position at time t_o…

was created one period of time, *T*, earlier than the voltage that is being applied to the sending-end at time t_o.

Based on this concept, the incident voltage seen at some distance, ***x***, from the sending-end of the line is the direct result of some voltage that was applied by the source to the sending-end of the line at an earlier point in time.

70

Phase of the Incident Voltage

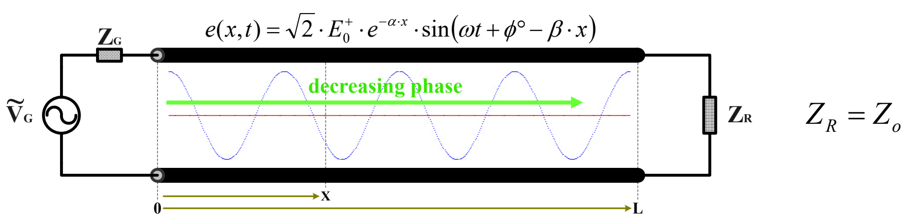


And since the **phase** of the voltage is directly **related to** the **time** at which that voltage was initially applied by the source to the sending-end of the line, then compared to the voltage being applied to the sending-end at some reference time $\boldsymbol{t_o}$, any voltage that was created earlier in time should have a phase that is smaller in value.

In other words, in order to account for the fact that the voltages seen further down the line were created earlier in time, the **phase** of those voltages **decreases as distance *x* increases**.

71

Phase Constant β



Thus, the **phase constant** term, $-\beta \cdot x$, in the expression:

$$e(x,t) = \sqrt{2} \cdot E_0^+ \cdot e^{-\alpha \cdot x} \cdot \sin(\omega t + \phi° - \beta \cdot x)$$

accounts for the **decrease in the phase** of the waveform as a as the distance ***x*** traveled from the sending-end of the line increases, which is a direct result of the finite propagation velocity of the wave on the line.

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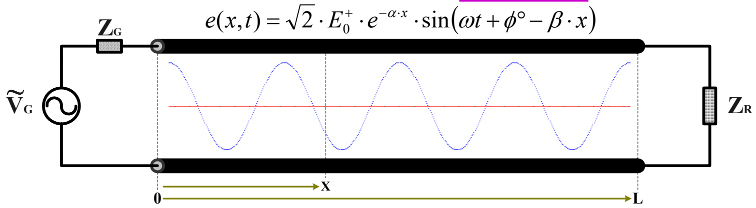
Phase Constant β

phase

$$e(x,t) = \sqrt{2} \cdot E_0^+ \cdot e^{-\alpha \cdot x} \cdot \sin(\omega t + \phi° - \beta \cdot x)$$



$Z_R = Z_o$

Phase constant, $\boldsymbol{\beta}$, can thus be related to the **phase velocity**, $\boldsymbol{v_p}$, of a wave that is propagating on a line by observing the change in the position of a constant point of phase as time increases.

And since the **phase** of the incident waveform is value "inside" the sine term when the waveform is expressed as a time-function, a **constant point of phase** occurs when:

$$(\omega t + \phi° - \beta \cdot x) = \text{constant}$$

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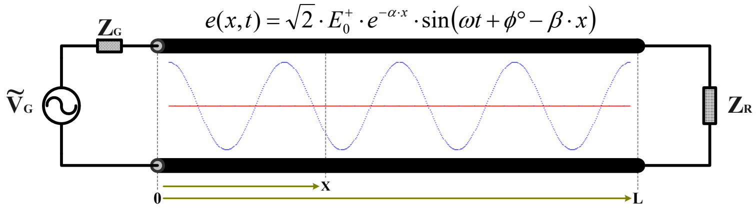
Phase Constant β

$$e(x,t) = \sqrt{2} \cdot E_0^+ \cdot e^{-\alpha \cdot x} \cdot \sin(\omega t + \phi° - \beta \cdot x)$$



$Z_R = Z_o$

By taking the derivative w.r.t. time of both sides of the expression:

$$(\omega t + \phi° - \beta \cdot x) = \text{constant}$$

we get:

$$\frac{d}{dt}(\omega t + \phi° - \beta \cdot x) = \frac{d}{dt}(\text{constant})$$

which can be simplified, as shown on the next slide.

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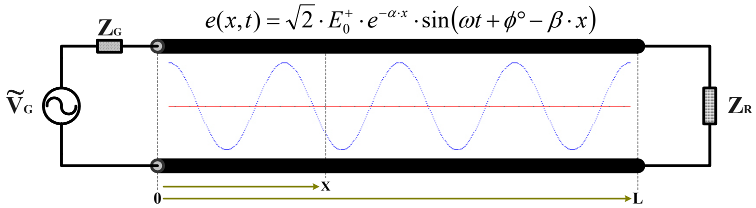
Phase Constant β

$$e(x,t)=\sqrt{2}\cdot E_0^+\cdot e^{-\alpha\cdot x}\cdot\sin(\omega t+\phi^\circ-\beta\cdot x)$$



$Z_R = Z_o$

Simplifying the expression:

$$\frac{d}{dt}(\omega t+\phi^\circ-\beta\cdot x)=\frac{d}{dt}(\text{constant})$$

$$\frac{d}{dt}(\omega t)+\frac{d}{dt}(\phi^\circ)-\frac{d}{dt}(\beta\cdot x)=0$$

$$\omega\frac{dt}{dt}+0-\beta\frac{dx}{dt}=0$$

$$\boxed{\omega-\beta\frac{dx}{dt}=0}$$

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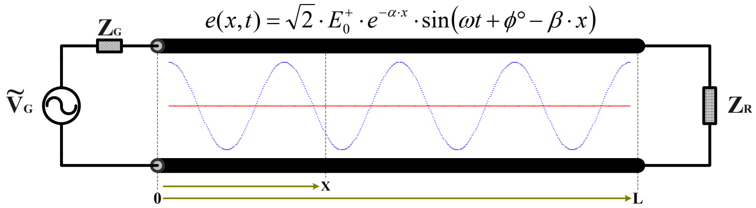
Phase Constant β

$$e(x,t)=\sqrt{2}\cdot E_0^+\cdot e^{-\alpha\cdot x}\cdot\sin(\omega t+\phi^\circ-\beta\cdot x)$$



$Z_R = Z_o$

But, given the resultant equation:

$$\omega-\beta\frac{dx}{dt}=0$$

$\frac{dx}{dt}$ is the **phase velocity**, v_p, which is the velocity of a constant point of phase on the line (i.e. – the velocity of the wave), allowing for the expression to be rewritten as:

$$\omega-\beta\cdot v_p=0$$

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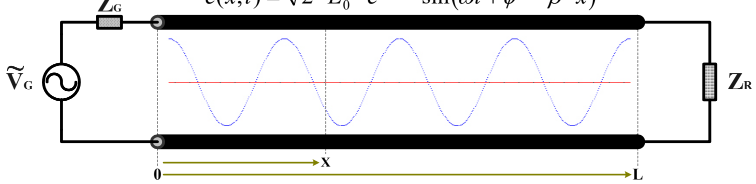
Phase Constant β

$$e(x,t) = \sqrt{2} \cdot E_0^{+} \cdot e^{-\alpha \cdot x} \cdot \sin(\omega t + \phi° - \beta \cdot x)$$



$Z_R = Z_o$

And based on the equation:

$$\omega - \beta \cdot v_p = 0$$

it can be seen that **phase velocity**: $v_p = \dfrac{\omega}{\beta}$

or that the **phase constant**: $\beta = \dfrac{\omega}{v_p}$

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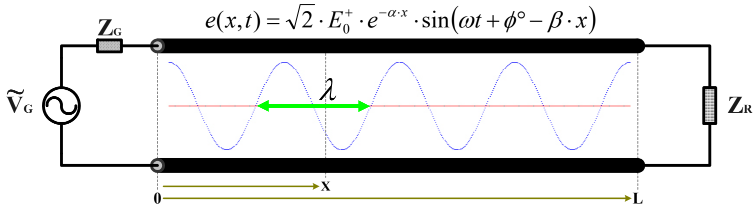
Wavelength λ vs. Phase Constant β

$$e(x,t) = \sqrt{2} \cdot E_0^{+} \cdot e^{-\alpha \cdot x} \cdot \sin(\omega t + \phi° - \beta \cdot x)$$



$Z_R = Z_o$

Note that the **wavelength**, λ, of the wave on the transmission line can then be expressed in terms of the phase constant as follows:

$$\beta = \frac{\omega}{v_p} = \frac{2\pi \cdot f}{v_p} = \frac{2\pi}{\left(\frac{v_p}{f}\right)} = \frac{2\pi}{\lambda} \quad \Rightarrow \quad \lambda = \frac{2\pi}{\beta}$$

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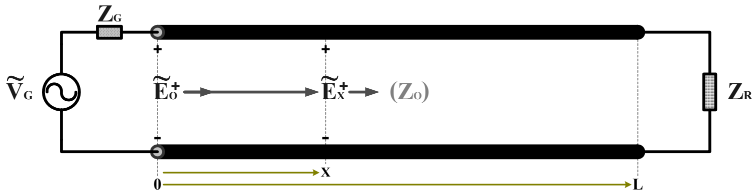
Characterizing the Attenuation Constant



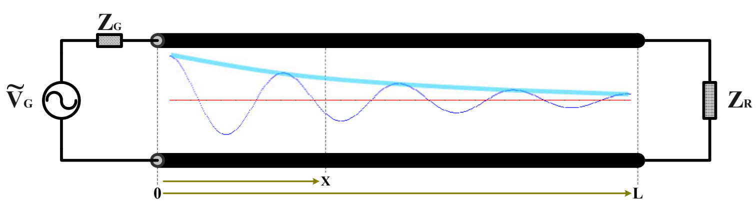
$Z_R = Z_o$

Now that we've characterized the phase constant by relating it to the decrease in the phase of a wave as it propagates on a line due to the finite propagation velocity, now lets go back and take a look at the effects of the **attenuation constant** on a lossy line.

magnitude affected by *α*

$$\tilde{E}_x^+ = \tilde{E}_o^+ \cdot e^{-\gamma \cdot x} = E_0^+ \cdot e^{-\alpha \cdot x} \angle \phi° - \beta \cdot x \text{ volts}$$

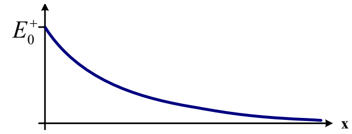
Attenuation on a Lossy Line



$Z_R = Z_o$

For a practical (**lossy**) line, the **attenuation constant** will be greater than zero (***α* >*0***), resulting in an **exponential decrease** in the **magnitude** of the **incident wave** as it travels down the line due to the exponential decay function $e^{-\alpha \cdot x}$.

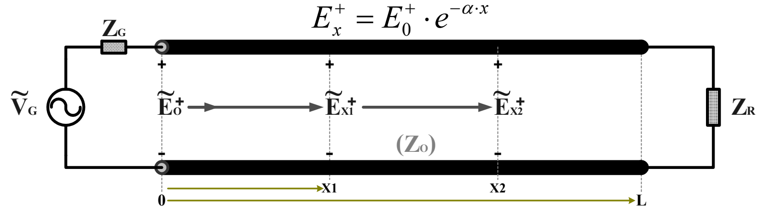
$$E_x^+ = E_0^+ \cdot e^{-\alpha \cdot x}$$



$$e(x,t) = \sqrt{2} \cdot E_0^+ \cdot e^{-\alpha \cdot x} \cdot \sin(\omega t + \phi° - \beta \cdot x)$$

Attenuation Constant α

$$E_x^+ = E_0^+ \cdot e^{-\alpha \cdot x}$$



The **attenuation constant** (***α***) for a line can be characterized in terms of the decrease in magnitude of a wave as it travels from position x_1 to position x_2 on the line. If expressed as a ratio of voltages:

$$\frac{E_{x2}^+}{E_{x1}^+} = \frac{E_0^+ \cdot e^{-\alpha \cdot x2}}{E_0^+ \cdot e^{-\alpha \cdot x1}} = \frac{e^{-\alpha \cdot x2}}{e^{-\alpha \cdot x1}} = e^{-\alpha \cdot (x2 - x1)} = e^{-\alpha \Delta x}$$

then: α is the **attenuation constant** in **nepers/meters**, and
Δx is the **distance traveled** in **meters**.

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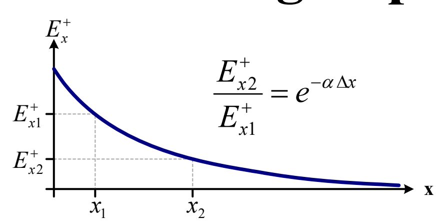
Characterizing Nepers



Note that the term $\alpha\Delta x$, which has the units of **nepers**, defines the **rate of attenuation** of the waveform's **magnitude** as the wave travels a specific distance Δx down the line.

You probably haven't heard of the term **nepers** before, but there is another term associated with attenuation that you probably have heard, and that term is **decibels**.

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Decibel (dB) Attenuation

Decibel attenuation is utilized to define the decrease in the power of a waveform (signal) at one point in a system compared to the power of a waveform (signal) at another point in the system.

Specifically, **decibel attenuation** is defined in terms of a **logarithmic ratio** of two powers:

$$\mathrm{dB}_{\mathrm{atten}} = -10 \log_{10} \frac{P_{x2}^{+}}{P_{x1}^{+}}$$

If $x = \log_{10} A$ **Then** $A = 10^{x}$

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Why Logarithms & Decibel Attenuation

$$\mathrm{dB}_{\mathrm{atten}} = -10 \log_{10} \frac{P_2}{P_1}$$

There are many types of measurements that can result in a dataset that covers an extremely wide range of values.

For example, the **sound pressure amplitude** of a jet at takeoff can be 3,000,000 times greater than the sound pressure amplitude of the quietest sound that a human can hear.

Examples of other measurements that can result in an extremely wide-ranging dataset include earthquake intensity (Richter scale), RF signal strength, and pH balance.

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Why Logarithms & Decibel Attenuation

$$\text{dB}_{\text{atten}} = -10\log_{10}\frac{P_2}{P_1}$$

Although simultaneously dealing with very large and very small numbers can be problematic, often making it very difficult to comprehend important details contained within the data, when expressed in terms of **decibels** (**logarithms)**, the wide-ranging dataset can be is compressed into something that is much more manageable.

For example, the aforementioned sound pressure amplitudes, that can differ by a factor of 3,000,000 from small to large, will only span the **range from 0 to 130** when expressed in **decibels**.

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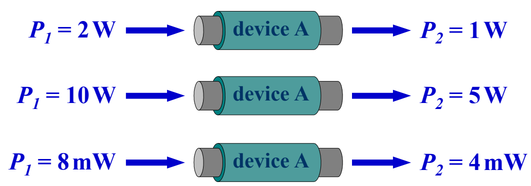
Why Decibel Attenuation

$$\text{dB}_{\text{atten}} = -10\log_{10}\frac{P_2}{P_1}$$

In high-frequency systems, there are many devices for which the **power loss** that occurs in a wave as it passes though the device is **based on a percentage** of the initial waveform's power (and not an exact power value), making them suitable for **dB** characterization.

For example: If "**device A**" cuts the signal **power** in half, then:

Note that, although the exact **power loss value** is **different in each case** (1 W vs. 5 W vs. 4 mW), **P_2 is always ½ (50%) of P_1.**



Practical (**lossy**) transmission lines function in this manner.

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Characteristics of Decibel Attenuation

And if the power, $\boldsymbol{P_2}$, of a signal as it exits a device is always ½ of the power, $\boldsymbol{P_1}$, of that same signal as it was entering the device, then the **decibel attenuation** of the signal caused by the device is:

$$\text{dB}_{\text{atten}} = -10\log_{10}\frac{P_2}{P_1} = -10\log_{10}\frac{\frac{1}{2}P_1}{P_1} = -10\log_{10}\frac{1}{2} \approx -10(-0.30) = \boxed{3\text{dB}}$$

3dB attenuation is equivalent to a **decrease or loss of power by a factor of ½.**
This is worth memorizing because it is a commonly referenced value.

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Characteristics of Decibel Attenuation

It turns out that every time $\boldsymbol{P_1}$ is **decreased by** an additional **factor of ½**, the **decibel attenuation increases by +3dB**.

P_2	dB Attenuation
$\frac{1}{2}P_1$	3dB
$\frac{1}{4}P_1$	6dB
$\frac{1}{8}P_1$	9dB
$\frac{1}{16}P_1$	12dB
$\frac{1}{32}P_1$	15dB

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Characteristics of Decibel Attenuation

It also turns out that every time P_1 is **decreased by a factor of 1⁄10**, the **decibel attenuation increases by +10dB**.

P_2	dB Attenuation
$\frac{1}{10}P_1$	**10dB**
$\frac{1}{100}P_1$	**20dB**
$\frac{1}{1000}P_1$	**30dB**
$\frac{1}{10000}P_1$	**40dB**
$\frac{1}{100000}P_1$	**50dB**

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Characteristics of Decibel Attenuation

But, if $P_2 = P_1$ (i.e. – there is no loss), then the **decibel attenuation** is:

$$\text{dB}_{\text{atten}} = -10\log_{10}\frac{P_2}{P_1} = -10\log_{10}\frac{P_1}{P_1} = -10\log_{10}1 \approx -10(0) = \boxed{0\text{dB}}$$

Intuitively this should make sense because, if decibel attenuation relates to a percent decrease in power, **0dB attenuation** relates to a **0% decrease**, and thus P_2 must equal P_1.

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Characteristics of Decibel Attenuation

Note that, if **original power**, P_1, and **decibel attenuation** are both known, the **remaining power**, P_2, can be calculated as follows:

$$\mathrm{dB}_{\mathrm{atten}} = -10\log_{10}\frac{P_2}{P_1} \quad \Rightarrow \quad \frac{\mathrm{dB}_{\mathrm{atten}}}{-10} = \log_{10}\frac{P_2}{P_1}$$

and since $A = 10^x$ **if** $x = \log_{10} A$

$$\frac{\mathrm{dB}_{\mathrm{atten}}}{-10} = \log_{10}\frac{P_2}{P_1} \quad \Rightarrow \quad \frac{P_2}{P_1} = 10^{\frac{\mathrm{dB}_{\mathrm{atten}}}{-10}} \quad \therefore \quad \boxed{P_2 = P_1 \cdot 10^{\frac{\mathrm{dB}_{\mathrm{atten}}}{-10}}}$$

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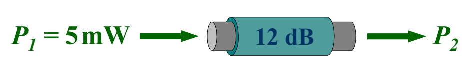
Decibel Calculation Examples

If the **signal power entering** an attenuator is measured to be **2.6 W**, and the **signal power exiting** the attenuator is measured to be **0.4 W**,

Determine the **decibel attenuation**.

$$\begin{aligned}\mathrm{dB}_{\mathrm{atten}} &= -10\log_{10}\frac{P_2}{P_1}\\ &= -10\log_{10}\frac{0.4}{2.6}\\ &= -10\log_{10}(0.15385)\\ &= -10(-0.813) = \boxed{8.13\,\mathrm{dB}}\end{aligned}$$



If the **signal power entering** a **12 dB attenuator** is measured to be **5 mW**,

Determine the **signal power** that will exit the attenuator.

$$\begin{aligned}P_2 &= P_1 \cdot 10^{\frac{\mathrm{dB}_{\mathrm{atten}}}{-10}}\\ &= (5\,\mathrm{mW}) \cdot 10^{\left(\frac{12}{-10}\right)}\\ &= (5\,\mathrm{mW}) \cdot 10^{(-1.2)}\\ &= (5\,\mathrm{mW}) \cdot (0.0631)\\ &= \boxed{0.3155\,\mathrm{mW}}\end{aligned}$$

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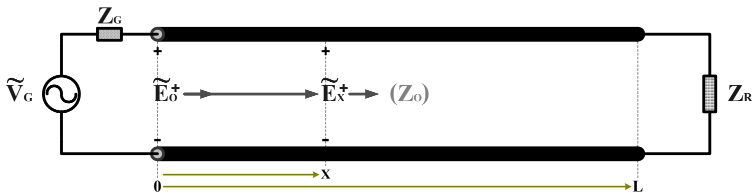
Relating Nepers to Decibels



Assuming that an incident voltage is propagating on a transmission line whose characteristic impedance is purely real, then the **power** associated with the **incident waveform**, $\boldsymbol{P_x^+}$, at position $\boldsymbol{x}$ on the line is equal to:

Incident Power →

$$P_x^+ = E_x^+ \cdot I_x^+ = E_x^+ \cdot \frac{E_x^+}{Z_o} = \frac{{E_x^+}^2}{Z_o}$$

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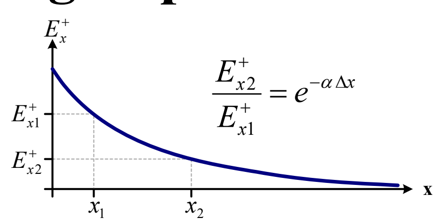
Relating Nepers to Decibels



And since incident power, $\boldsymbol{P_x^+}$, at position $\boldsymbol{x}$ is proportional to $\boldsymbol{(E_x^+)^2}$, **decibel attenuation** can also be defined in terms of a logarithmic ratio of **voltages**:

$$\text{dB}_{\text{atten}} = -10\log_{10}\frac{P_{x2}^+}{P_{x1}^+} = -10\log_{10}\frac{\frac{{E_{x2}^+}^2}{Z_o}}{\frac{{E_{x1}^+}^2}{Z_o}} = -10\log_{10}\left(\frac{E_{x2}^+}{E_{x1}^+}\right)^2 = \boxed{-20\log_{10}\left(\frac{E_{x2}^+}{E_{x1}^+}\right)}$$

Note: $\log_{10} a^b = b\log_{10} a$

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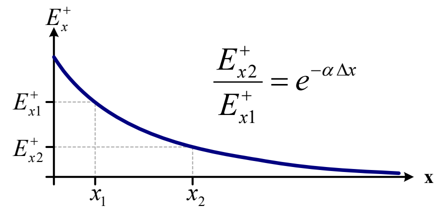
Relating Nepers to Decibels



$$\frac{E_{x2}^+}{E_{x1}^+} = e^{-\alpha\,\Delta x}$$

If the (neper) ratio of voltages is substituted into the expression for decibels:

$$\frac{E_{x2}^+}{E_{x1}^+} = e^{-\alpha\,\Delta x} \quad \Longrightarrow \quad \text{dB}_{\text{atten}} = -20\log_{10}\frac{E_{x2}^+}{E_{x1}^+}$$

Then the following relationship between **nepers** and $\mathbf{dB_{atten}}$ can be derived:

$$\text{dB}_{\text{atten}} = -20\log_{10} e^{-\alpha\,\Delta x}$$

$$= \alpha\,\Delta x \cdot 20\log_{10} e$$

$$\boxed{\text{dB}_{\text{atten}}} = \alpha\,\Delta x \cdot 8.686 = \boxed{\text{Nepers} \cdot 8.686}$$

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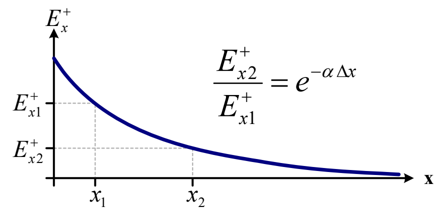
Relating Nepers to Decibels



$$\frac{E_{x2}^+}{E_{x1}^+} = e^{-\alpha\,\Delta x}$$

Thus, **neper attenuation** is related to **decibel attenuation** as follows:

$$\text{Nepers} = \frac{\text{dB}}{8.686} = 0.11513 \cdot \text{dB}$$

$$\text{dB} = 8.686 \cdot \text{Nepers}$$

Note that, since **decibel attenuation** information for standard cable types is provided in **Table 1.3**, that information can be used to determine the **attenuation constant**, α, for those cable types.

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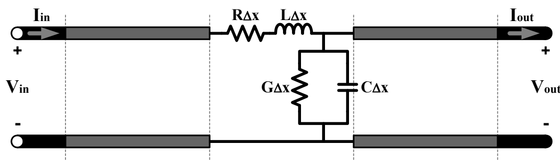
Determining the Propagation Constant γ



Although we originally derived the expression for the **propagation constant**, γ, from the parameters that composed the incremental model of the transmission line:

$$\gamma = \alpha + j\beta = \sqrt{Z \cdot Y} = \sqrt{(R + j\omega L)(G + j\omega C)}$$

in the vast majority of cases, you will **not** have knowledge of the values of those parameters.

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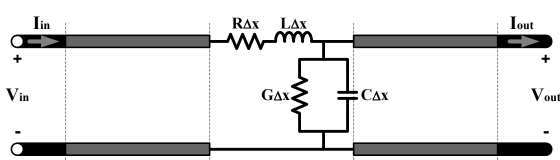
Determining the Propagation Constant γ



But, instead of trying to determine the value of the **propagation constant**, γ, from the parameters, it turns out that you can solve the individual components $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in the expression for $\boldsymbol{\gamma = \alpha + j\beta}$, for a variety of common types of coaxial cables, directly from the information provided in Table 1-3 of the textbook.

Coaxial Cables											
RG #	AWG Material	Insulation	# Shields	Jacket	Nom. O.D. (inch)	Nom. Imp. (Ohms)	Nom. Vel. Of Prop.	Nom. Cap. (pF/ft.)	Nom. Attenuation per 100' MHz	dB	Standard Spool Lengths
14/U	20 Copper	Poly-ethylene	1	Black Vinyl	.420	95	66%	16.0	100 200 400	3.0 4.5 6.0	100, 500

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Determining the Attenuation Constant α

Coaxial Cables											
RG #	AWG Material	Insulation	# Shields	Jacket	Nom. O.D. (inch)	Nom. Imp. (Ohms)	Nom. Vel. Of Prop.	Nom. Cap. (pF/ft.)	Nom. Attenuation per 100' MHz	dB	Standard Spool Lengths
14/U	20 Copper	Poly-ethylene	1	Black Vinyl	.420	95	66%	16.0	100 200 400	3.0 4.5 6.0	100, 500

Determine the **attenuation constant α** for RG 14/U cable at 100MHz.

$$\text{dB}_{\text{atten}} = 3.0 \tfrac{\text{dB}}{100\text{feet}} \cdot \frac{1}{100} = 0.03 \tfrac{\text{dB}}{\text{foot}} \cdot 3.281 \tfrac{\text{feet}}{\text{meter}} = 0.09843 \tfrac{\text{dB}}{\text{meter}}$$

Thus:

$$\alpha \tfrac{\text{Nepers}}{\text{meter}} = 0.09843 \tfrac{\text{dB}}{\text{meter}} \cdot 0.11513 \tfrac{\text{Nepers}}{\text{dB}} = \boxed{0.01133 \tfrac{\text{Nepers}}{\text{meter}}}$$

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Determining the Phase Constant β

Coaxial Cables											
RG #	AWG Material	Insulation	# Shields	Jacket	Nom. O.D. (inch)	Nom. Imp. (Ohms)	Nom. Vel. Of Prop.	Nom. Cap. (pF/ft.)	Nom. Attenuation per 100' MHz	dB	Standard Spool Lengths
14/U	20 Copper	Poly-ethylene	1	Black Vinyl	.420	95	66%	16.0	100 200 400	3.0 4.5 6.0	100, 500

Determine the **phase constant β** for RG 14/U cable at 100MHz.

$$\text{Nom. Vel. of Prop.} = 66\%$$

Therefore:

$$v_p = (0.66) \cdot \text{c} = (0.66) \cdot (3\text{x}10^8) = 1.98\text{x}10^8 \tfrac{\text{meters}}{\text{second}}$$

$$\beta = \frac{\omega}{v_p} = \frac{2\pi \cdot 100\text{x}10^6}{1.98\text{x}10^8} = \boxed{3.173 \tfrac{\text{radians}}{\text{meter}}}$$

100

Determining the Propagation Constant γ

Coaxial Cables											
RG #	AWG Material	Insulation	# Shields	Jacket	Nom. O.D. (inch)	Nom. Imp. (Ohms)	Nom. Vel. Of Prop.	Nom. Cap. (pF/ft.)	Nom. Attenuation per 100' MHz	dB	Standard Spool Lengths
14/U	20 Copper	Poly-ethylene	1	Black Vinyl	.420	95	66%	16.0	100 200 400	3.0 4.5 6.0	100, 500

Based on the previous results, the **propagation constant γ** for **RG 14/U** cable at **100MHz** is:

$$\gamma = \alpha + j\beta = 0.0113315 \frac{Nepers}{meter} + j3.173 \frac{\text{radians}}{\text{meter}}$$

which can be used to solve for the value of the incident voltage as a function of position provided that the value of the incident voltage $\tilde{\boldsymbol{E}}_o^{\,+}$ applied to the sending-end of the cable is known:

$$\tilde{E}(x) = \tilde{E}_o^{+} \cdot e^{-\gamma \cdot x}$$

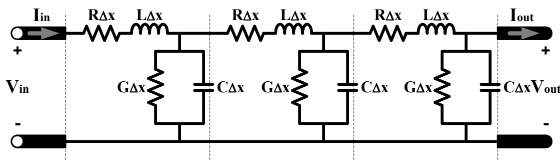
101

AC Transmission Lines
Part IV

Traveling AC Waves
on an
Unmatched Transmission Line

102

Steady-State Voltage on a Transmission Line



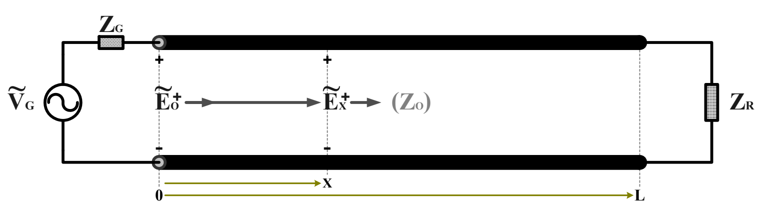
In **part I** of this presentation, we developed an incremental model for a transmission-line and then utilized that model to derive the **general solution** for **steady-state voltage** on the transmission line as a function of position, ***x***:

$$\widetilde{E}(x) = \widetilde{E}_x^+ + \widetilde{E}_x^- = A_1 \cdot e^{-\gamma \cdot x} + A_2 \cdot e^{+\gamma \cdot x}$$

the first term of which defines the **incident voltage** on the line, while the second term defines the **reflected voltage** on the line.

103

Solving for the Constant A_1



$Z_R = Z_o$

In **part II** of this presentation, we considered the case of a matched transmission-line in order to isolate the first term in the general solution that related to the **incident voltage** on the line:

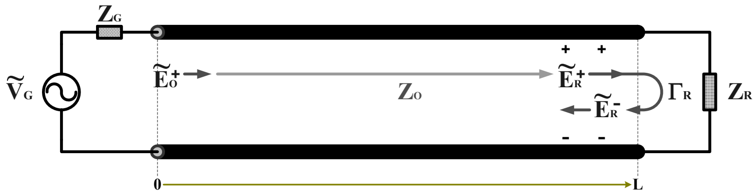
$$\widetilde{E}_x^+ = A_1 \cdot e^{-\gamma \cdot x}$$

and determined an expression for the **constant A_1**, where:

$A_1 = \widetilde{E}_o^+$ (the incident voltage applied to the sending-end by the source)

104

Solving for the Constant A_2



$Z_R \neq Z_o$

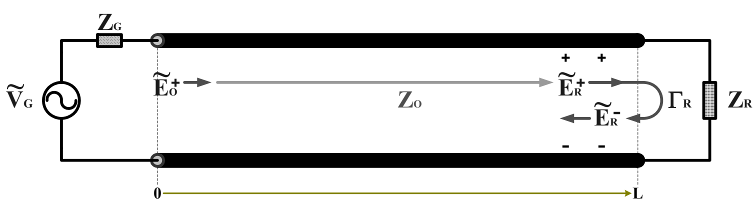
If we substitute this result into the equation for the general solution of **steady-state voltage** on the transmission line as a function of position, ***x***, we are then left with an equation that still contains the **constant A_2** in the term that defines the reflected voltage:

$$\widetilde{E}(x) = \widetilde{E}_x^+ + \widetilde{E}_x^- = \widetilde{E}_o^+ \cdot e^{-\gamma \cdot x} + \boxed{A_2 \cdot e^{+\gamma \cdot x}}$$

In order to solve for this constant, we will now consider the case of an **unmatched** transmission line ($Z_R \neq Z_o$).

105

Reflected Voltage on an Unmatched Line



$Z_R \neq Z_o$

If the load impedance is **not** matched to the characteristic impedance of the line:

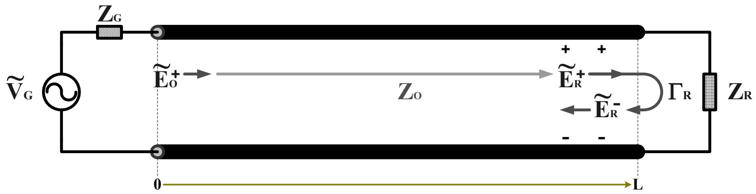
$$Z_R \neq Z_o$$

then a **reflection** $\widetilde{E}_R^-$ will occur when the incident wave reaches the receiving-end of the line, such that:

$$\widetilde{E}_R^- = \widetilde{E}_R^+ \cdot \Gamma_R \qquad \text{where} \qquad \Gamma_R = \frac{Z_R - Z_o}{Z_R + Z_o}$$

106

Reflected Voltage on an Unmatched Line



$Z_R \neq Z_o$

And if a reflection occurs, the **reflection term** $\tilde{E}_x^-$ in the solution:

$$\tilde{E}(x) = \tilde{E}_x^+ + \boxed{\tilde{E}_x^-} = \tilde{E}_o^+ \cdot e^{-\gamma \cdot x} + \boxed{A_2 \cdot e^{+\gamma \cdot x}}$$

will no longer be zero, allowing us to solve for the constant A_2 in the same manner for which we determined the constant A_1.

Note that the reflection produced by the unmatched load will experience the same propagation effects as it travels in the "-*x*" direction back towards the sending-end of the line.

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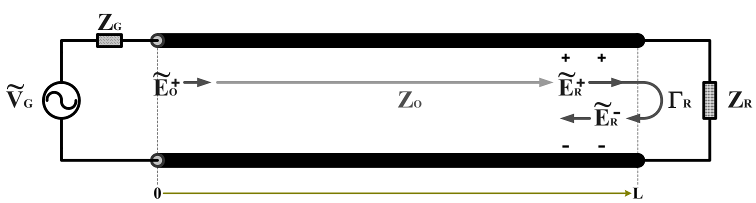
Reflected Voltage on an Unmatched Line



$Z_R \neq Z_o$

Since the applied incident waveform must travel distance ***L*** to reach the receiving-end of the line, the value of the **incident wave** that reaches the **receiving-end** (***x*=*L***) is:

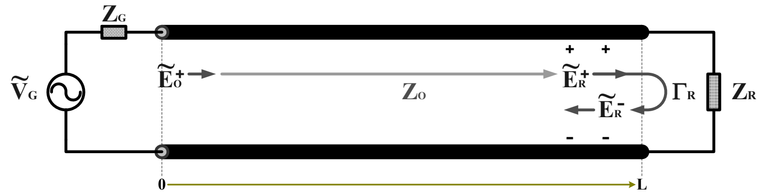
$$\tilde{E}_R^+ = \tilde{E}_0^+ \cdot e^{-\gamma \cdot L}$$

resulting in the creation of a **reflected wave** at the **receiving-end** that is equal to:

$$\tilde{E}_R^- = \tilde{E}_R^+ \cdot \Gamma_R = \tilde{E}_0^+ \cdot \Gamma_R \cdot e^{-\gamma \cdot L}$$

108

Solving for the Constant A_2



$Z_R \neq Z_o$

By substituting $\boldsymbol{x}=\boldsymbol{L}$ into the general solution for **reflected voltage** and setting it equal to the known value of the **reflected voltage** at the **receiving-end** that we defined in terms of the incident voltage:

$$\tilde{E}_R^- = A_2 \cdot e^{+\gamma \cdot L} = \tilde{E}_0^+ \cdot \Gamma_R \cdot e^{-\gamma \cdot L}$$

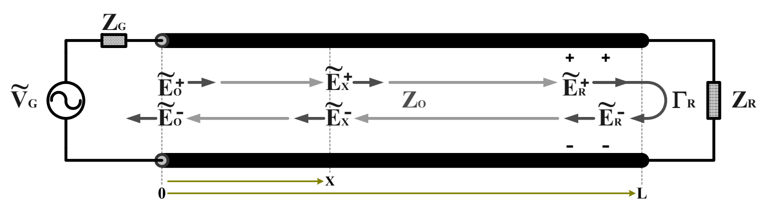
the **constant A_2** can be determined as follows:

$$A_2 = \frac{\tilde{E}_0^+ \cdot \Gamma_R \cdot e^{-\gamma \cdot L}}{e^{+\gamma \cdot L}} = \boxed{\tilde{E}_0^+ \cdot \Gamma_R \cdot e^{-2 \cdot \gamma \cdot L}}$$

This equates to the incident wave applied to the sending-end after it travels to and reflects off the load.

109

Steady-State Voltage on an Unmatched Line



$$\Gamma_R = \frac{Z_R - Z_o}{Z_R + Z_o}$$

Now that constants A_1 and A_2 are known, the **steady-state** solution for **voltage** on a transmission-line as a **function of position**, $\boldsymbol{x}$, is:

$$\tilde{E}(x) = \tilde{E}_x^+ + \tilde{E}_x^- = \boxed{\tilde{E}_0^+ \cdot e^{-\gamma \cdot x} + \tilde{E}_0^+ \cdot \Gamma_R \cdot e^{-2 \cdot \gamma \cdot L} \cdot e^{+\gamma \cdot x}}$$

where: $\tilde{E}_0^+$ is the phasor value of the **sending-end incident voltage**,
Γ_R is the value of the **reflection coefficient** due to the load,
γ is the **propagation constant** for the line, and
L is the **length** of the line.

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Steady-State Voltage on an Unmatched Line

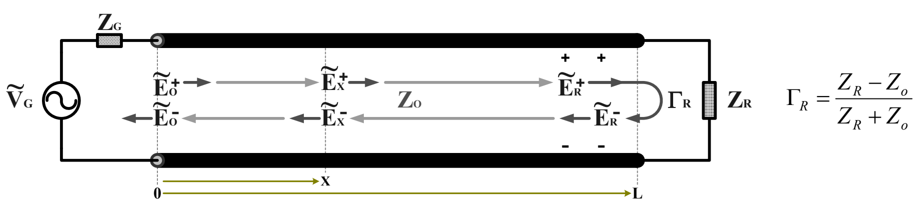
Now that constants A_1 and A_2 are known, the **steady-state** solution for **voltage** on a transmission-line as a **function of position**, x, is:

$$\tilde{E}(x) = \tilde{E}_x^+ + \tilde{E}_x^- = \tilde{E}_0^+ \cdot e^{-\gamma \cdot x} + \tilde{E}_0^+ \cdot \Gamma_R \cdot e^{-2\cdot\gamma\cdot L} \cdot e^{+\gamma\cdot x}$$

Keep in mind that the terms shown in this equation define the incident and reflected voltages when expressed as their phasor equivalents, and that the incident and reflected voltages are actually sinusoidally-varying waveforms that are traveling in opposite directions on the line.

111

Steady-State Voltage on an Unmatched Line



Based on the previous solution, the **steady-state voltage** at the **sending-end** of the line (***x=0***) is:

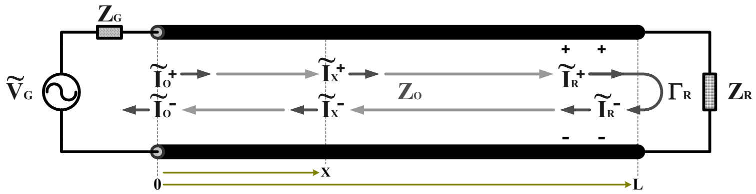
$$\tilde{E}_S = \tilde{E}(0) = \tilde{E}_0^+ \cdot e^{-\gamma\cdot 0} + \tilde{E}_0^+ \cdot \Gamma_R \cdot e^{-2\cdot\gamma\cdot L} \cdot e^{+\gamma\cdot 0} = \tilde{E}_0^+ + \tilde{E}_0^+ \cdot \Gamma_R \cdot e^{-2\cdot\gamma\cdot L}$$

The first term is the incident voltage that the source applies to the sending-end of the line.

The second term is the result of the sending-end incident voltage after it travels to the receiving-end, reflects off the load, and travels back to the sending-end of the line.

112

Steady-State Current on an Unmatched Line



The **steady-state** solution for **current** on a transmission-line as a **function of position** can be defined in terms of the voltage as:

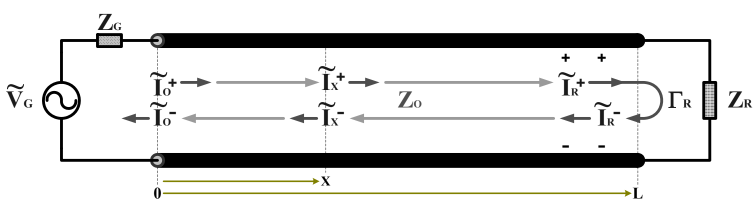
$$\tilde{I}(x) = \frac{\tilde{E}_0^+ \cdot e^{-\gamma \cdot x}}{Z_o} - \frac{\tilde{E}_0^+ \cdot \Gamma_R \cdot e^{-2\cdot\gamma\cdot L} \cdot e^{+\gamma\cdot x}}{Z_o}$$

since:

$$\tilde{I}(x) = \tilde{I}_x^+ - \tilde{I}_x^- = \frac{\tilde{E}_x^+}{Z_o} - \frac{\tilde{E}_x^-}{Z_o}$$

113

Steady-State Current on an Unmatched Line



In-turn, the **steady-state** for **current** at the **sending-end** of the line (***x***=***0***) is:

$$\tilde{I}_S = \tilde{I}(0) = \frac{\tilde{E}_0^+}{Z_o} - \frac{\tilde{E}_0^+ \cdot \Gamma_R \cdot e^{-2\cdot\gamma\cdot L}}{Z_o}$$

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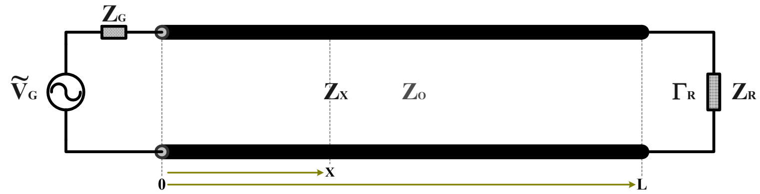
Impedance on an Unmatched Line



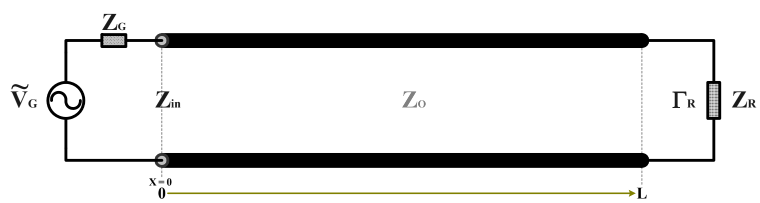
$Z_R \neq Z_o$

Since **impedance** on the line **at position *x*** can be defined as the ratio of the steady-state voltage and current at position *x*:

$$Z(x) = \frac{\tilde{E}(x)}{\tilde{I}(x)} = \frac{\tilde{E}_0^+ \cdot e^{-\gamma \cdot x} + \tilde{E}_0^+ \cdot \Gamma_R \cdot e^{-2\cdot\gamma\cdot L} \cdot e^{+\gamma\cdot x}}{\frac{\tilde{E}_0^+ \cdot e^{-\gamma\cdot x}}{Z_o} - \frac{\tilde{E}_0^+ \cdot \Gamma_R \cdot e^{-2\cdot\gamma\cdot L} \cdot e^{+\gamma\cdot x}}{Z_o}} = Z_o \cdot \frac{1 + \Gamma_R \cdot e^{-2\cdot\gamma\cdot(L-x)}}{1 - \Gamma_R \cdot e^{-2\cdot\gamma\cdot(L-x)}}$$

115

Input Impedance of a Transmission-Line



The result:

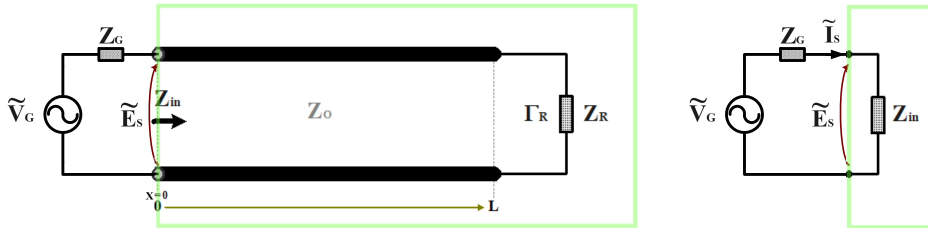
$$Z(x) = Z_o \cdot \frac{1 + \Gamma_R \cdot e^{-2\cdot\gamma\cdot(L-x)}}{1 - \Gamma_R \cdot e^{-2\cdot\gamma\cdot(L-x)}}$$

can be utilized to determine an expression for the **steady-state** (sending-end) **input impedance** of the line (***x*=*0***):

$$Z_{in(x=0)} = Z_o \cdot \frac{1 + \Gamma_R \cdot e^{-2\cdot\gamma\cdot L}}{1 - \Gamma_R \cdot e^{-2\cdot\gamma\cdot L}}$$

116

Input Impedance of a Transmission-Line



The **input impedance** of the line :

$$Z_{in} = Z_o \cdot \frac{1 + \Gamma_R \cdot e^{-2\cdot\gamma\cdot L}}{1 - \Gamma_R \cdot e^{-2\cdot\gamma\cdot L}}$$

is often of interest because it defines the **impedance** that the **source experiences** under **steady-state conditions**, and thus can be used as an equivalent impedance in place of the "line-load" combination, allowing a phasor-analysis of the system's operation from the perspective of the source.

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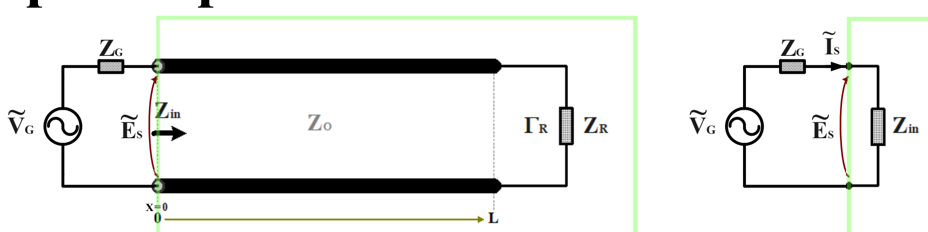
Input Impedance of a Transmission-Line



In other words, the **input impedance** of the line is equal to the ratio of the steady-state, sending-end, voltage and current:

$$Z_{in(x=0)} = \frac{\tilde{E}_S}{\tilde{I}_S}$$

and as such, if the "line-load" combination is replaced by an equivalent impedance $\boldsymbol{Z_{in}}$, then the source's operation can be analyzed by means of a phasor analysis of the simplified circuit.

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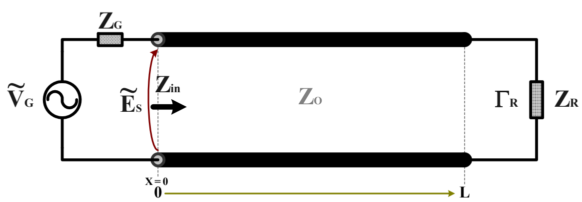
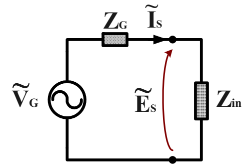
Input Impedance of a Transmission-Line





$$Z_{in} = \frac{\tilde{E}_S}{\tilde{I}_S}$$

For example, if the **input impedance** of the line is utilized, then the **steady-state current** can be solved using Ohm's Law and the **steady-state voltage** can be solved from a voltage-divider equation:

$$\tilde{I}_S = \frac{\tilde{V}_G}{Z_G + Z_{in}} \qquad \tilde{E}_S = \tilde{V}_G \cdot \frac{Z_{in}}{Z_G + Z_{in}}$$

This also allows for the use of other circuit theorems, such as the maximum power transfer theorem, which states that a source will deliver maximum power to a load if the load impedance is equal to the (purely real) series impedance of the source.

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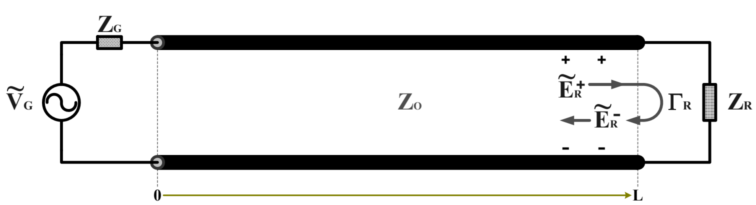
Reflection Coefficient vs. Load Impedance



The **reflection coefficient**, Γ_R, **of the load** was defined as the ratio of reflected and incident voltages at the receiving-end of the line:

$$\Gamma_R = \frac{\tilde{E}_R^-}{\tilde{E}_R^+} = \frac{Z_R - Z_o}{Z_R + Z_o}$$

From this, the **load impedance**, Z_R, can be defined in terms of reflection coefficient:

$$Z_R = Z_o \cdot \frac{1 + \Gamma_R}{1 - \Gamma_R}$$

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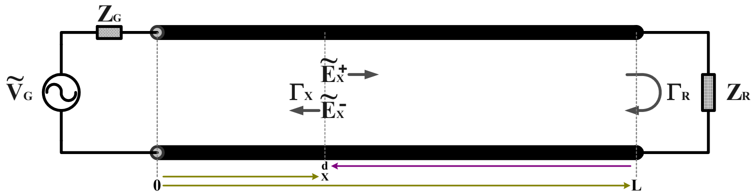
Reflection Coefficient on a Line



In addition to defining an impedance at any point on a transmission line as the ratio of the steady-state voltage and current at that position, a **reflection coefficient** can also be defined as a **ratio** of the **reflected and incident voltages** at that position:

$$\Gamma_X = \frac{\tilde{E}_X^-}{\tilde{E}_X^+}$$

even though the actual reflection occurs <u>at the receiving end</u>.

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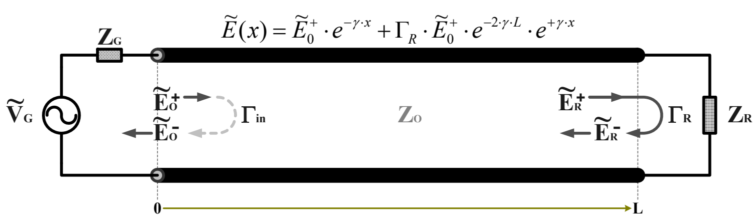
Input Reflection Coefficient Γ_{in}



And based on this concept, an **input reflection coefficient**, Γ_{in}, can also be defined as the ratio of the reflected and incident voltages at the sending-end of the line:

$$\Gamma_{in} = \frac{\tilde{E}_0^-}{\tilde{E}_0^+} = \frac{\tilde{E}_0^+ \cdot \Gamma_R \cdot e^{-2\cdot\gamma\cdot L}}{\tilde{E}_0^+} = \Gamma_R \cdot e^{-2\cdot\gamma\cdot L}$$

$$\boxed{\Gamma_{in} = \Gamma_R \cdot e^{-2\cdot\gamma\cdot L}}$$

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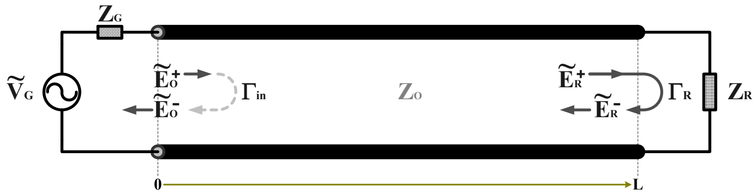
Reflection Coefficient vs. Input Impedance



Since:

$$\Gamma_{in} = \Gamma_R \cdot e^{-2\cdot\gamma\cdot L}$$

the expression for the **input impedance** can be rewritten as:

$$Z_{in} = Z_o \cdot \frac{1+\Gamma_R \cdot e^{-2\cdot\gamma\cdot L}}{1-\Gamma_R \cdot e^{-2\cdot\gamma\cdot L}} = \boxed{Z_o \cdot \frac{1+\Gamma_{in}}{1-\Gamma_{in}}}$$

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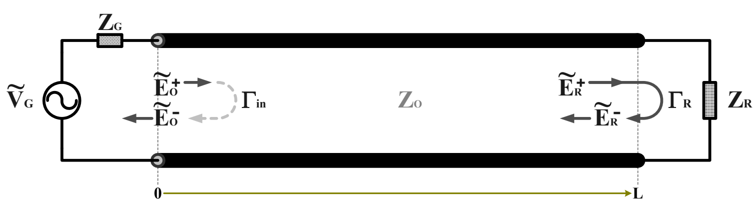
Reflection Coefficient and Impedance



Note that the expression for input impedance is similar in form to the expression for load impedance (in terms of reflection coefficient):

$$Z_{in} = Z_o \cdot \frac{1+\Gamma_{in}}{1-\Gamma_{in}} \qquad Z_R = Z_o \cdot \frac{1+\Gamma_R}{1-\Gamma_R}$$

This similarity, along with the relationship $\Gamma_{in} = \Gamma_R \cdot e^{-2\cdot\gamma\cdot L}$, provides the foundation upon which a **graphical solution** for impedance on a transmission-line using a Smith Chart will be derived*.

*** - in an upcoming lecture**

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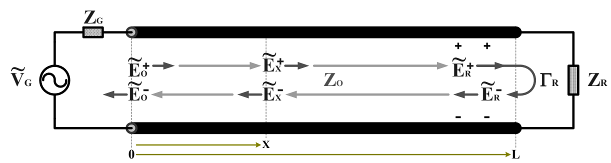
Example Problem



Given a **5.5 meter** section of **RG 14/U** supplied by a source that has the following characteristics:

$$\tilde{V}_G = 6\angle 0° \text{ volts} \qquad Z_G = 95\,\Omega \qquad f = 100 \text{ MHz}$$

determine the **steady-state voltage** at the **sending-end**, $\tilde{E}_S$, the **steady-state voltage** at the **receiving-end**, $\tilde{E}_R$, and the **input impedance** of the line, Z_{in}, if the line is terminated by a load having the impedance $Z_R = 200\,\Omega$.

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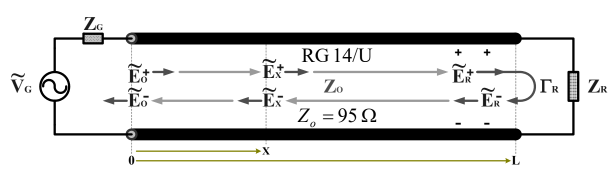
Example Problem

$\tilde{V}_G = 6\angle 0°$ volts

$Z_G = 95\,\Omega$

$f = 100$ MHz



$Z_R = 200\,\Omega$

Coaxial Cables

RG #	AWG Material	Insulation	# Shields	Jacket	Nom. O.D. (inch)	Nom. Imp. (Ohms)	Nom. Vel. Of Prop.	Nom. Cap. (pF/ft.)	Nom. Attenuation per 100' MHz	dB	Standard Spool Lengths
14/U	20 Copper	Poly-ethylene	1	Black Vinyl	.420	95	66%	16.0	100 200 400	3.0 4.5 6.0	100, 500

For **RG 14/U**, the **characteristic impedance** and **velocity** are:

$$Z_o = 95\,\Omega \qquad v = v\% \cdot c = (0.66)\cdot(3x10^8) = 1.98x10^8 \ \frac{\text{meters}}{\text{sec}}$$

thus wavelength at **<u>100MHz</u>** is:

$$\lambda = \frac{v}{f} = \frac{1.98x10^8}{100x10^6} = \boxed{1.98 \text{ meters}}$$

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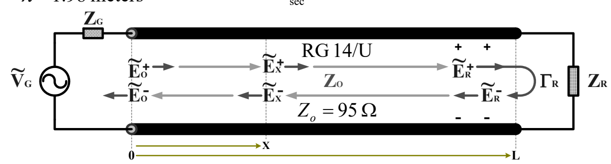
Example Problem

$\lambda = 1.98$ meters $\quad v = 1.98x10^8 \frac{\text{meters}}{\text{sec}}$

$\tilde{V}_G = 6\angle 0°$ volts

$Z_G = 95\,\Omega$

$f = 100$ MHz



$Z_R = 200\,\Omega$

Coaxial Cables											
RG #	AWG Material	Insulation	# Shields	Jacket	Nom. O.D. (inch)	Nom. Imp. (Ohms)	Nom. Vel. Of Prop.	Nom. Cap. (pF/ft.)	Nom. Attenuation per 100' MHz	dB	Standard Spool Lengths
14/U	20 Copper	Poly-ethylene	1	Black Vinyl	.420	95	66%	16.0	100 200 400	3.0 4.5 6.0	100, 500

Keep at least **4** **significant-digits** of accuracy throughout all calculations!

For **RG 14/U**, the **propagation constant** $\boldsymbol{\gamma}$ is:

$$\alpha \frac{\text{Nepers}}{\text{meter}} = (3.0 \frac{\text{dB}}{100\text{feet}}) \cdot (\frac{1}{100}) \cdot (3.281 \frac{\text{feet}}{\text{meter}}) \cdot (0.11513 \frac{\text{Nepers}}{\text{dB}}) = \boxed{0.01133 \frac{\text{Nepers}}{\text{meter}}}$$

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi \text{ rad}}{1.98 \text{ meters}} = \boxed{3.173 \frac{\text{radians}}{\text{meter}}}$$

$$\boxed{\gamma = \alpha + j\beta = 0.01133 \frac{Nepers}{meter} + j3.173 \frac{\text{radians}}{\text{meter}}}$$

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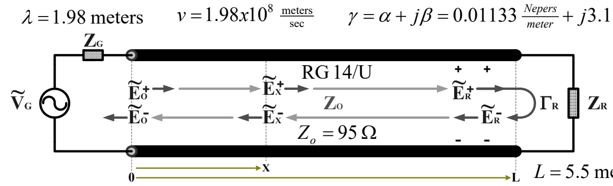
Example Problem

$\lambda = 1.98$ meters $\quad v = 1.98x10^8 \frac{\text{meters}}{\text{sec}} \quad \gamma = \alpha + j\beta = 0.01133 \frac{Nepers}{meter} + j3.173 \frac{\text{radians}}{\text{meter}}$

$\tilde{V}_G = 6\angle 0°$ volts

$Z_G = 95\,\Omega$

$f = 100$ MHz



$Z_R = 200\,\Omega$

$L = 5.5$ meters

The **reflection coefficient**, $\boldsymbol{\Gamma_R}$, due to the mismatched load is:

$$\Gamma_R = \frac{Z_R - Z_o}{Z_R + Z_o} = \frac{200 - 95}{200 + 95} = \boxed{0.356}$$

Note that although the resultant reflection coefficient is a real number, the reflection coefficient can be complex if the load impedance has a reactive component (inductive or capacitive).

and the **incident voltage** applied to the **sending-end**, $\tilde{E}_o^+$, is:

$$\tilde{E}_0^+ = \tilde{V}_G \cdot \left(\frac{Z_o}{Z_G + Z_o} \right) = (6\angle 0°) \cdot \left(\frac{95}{95 + 95} \right) = \boxed{3\angle 0° \text{ volts}}$$

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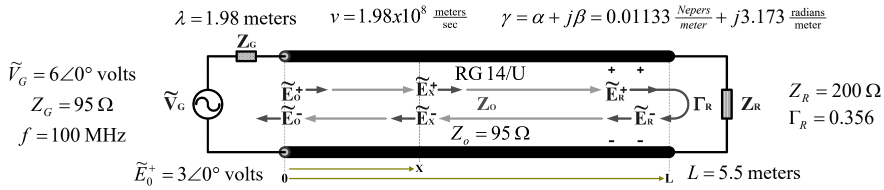
Example Problem



Thus, given the equation: $\tilde{E}_S = \tilde{E}_0^+ + \Gamma_R \cdot \tilde{E}_0^+ \cdot e^{-2\cdot\gamma\cdot L}$

since **x = 0** at the **sending-end** of the line

the steady-state **voltage** at the **sending-end** of the line, $\tilde{E}_S$, is:

$$\begin{aligned}\tilde{E}_S = \tilde{E}(0) &= (3\angle 0°) \ + \ (0.356)\cdot(3\angle 0°)\cdot e^{-2\cdot(0.01133+j3.173)\cdot(5.5)} \\ &= (3\angle 0°) \ + \ (0.356)\cdot(3\angle 0°)\cdot(-0.8307 + j0.2990) \\ &= (3\angle 0°) \ + \ (-0.8871 + j0.3193) \\ &= (2.113 + j0.3193) \ = \ (2.14\angle 0.150\,\text{rad}) \ = \ \boxed{(2.14\angle 8.6°)\ \text{volts}}\end{aligned}$$

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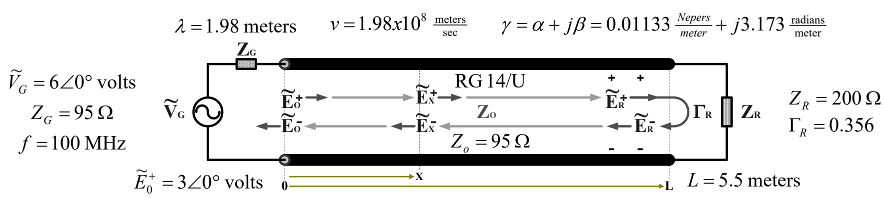
Example Problem



Note that the steady-state **current** at the **sending-end** of the line, $\tilde{I}_S$, is:

$$\begin{aligned}\tilde{I}_S = \tilde{I}_0^+ - \tilde{I}_0^- &= \frac{\tilde{E}_0^+}{Z_o} - \frac{\tilde{E}_0^-}{Z_o} = \frac{\tilde{E}_0^+}{Z_o} - \frac{\Gamma_R \cdot \tilde{E}_0^+ \cdot e^{-2\cdot\gamma\cdot L}}{Z_o} \\ &= \frac{(3\angle 0°)}{95} - \frac{(-0.8871 + j0.3193)}{95} \\ &= (0.03158\angle 0°) - (-0.00938 + j0.00336) \\ &= (0.041 - j0.00336) = (0.041\angle -0.082\,\text{rad}) = \boxed{(0.041\angle -4.70°)\ \text{amps}}\end{aligned}$$

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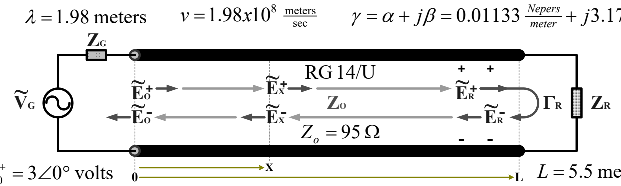
Example Problem

$\lambda = 1.98$ meters $\quad v = 1.98x10^8 \ \frac{\text{meters}}{\text{sec}} \quad \gamma = \alpha + j\beta = 0.01133 \ \frac{Nepers}{meter} + j3.173 \ \frac{\text{radians}}{\text{meter}}$

$\widetilde{V}_G = 6\angle 0°$ volts
$Z_G = 95\,\Omega$
$f = 100$ MHz



$\widetilde{E}_0^+ = 3\angle 0°$ volts

$Z_R = 200\,\Omega$
$\Gamma_R = 0.356$
$L = 5.5$ meters

x = 5.5 at the receiving-end of the line

And given the equation: $\widetilde{E}(x) = \widetilde{E}_0^+ \cdot e^{-\gamma \cdot x} + \Gamma_R \cdot \widetilde{E}_0^+ \cdot e^{-2\cdot\gamma\cdot L} \cdot e^{+\gamma\cdot x}$

the steady-state **voltage** at the **receiving-end** of the line, $\widetilde{E}_R$, is:

$$\begin{aligned}
\widetilde{E}_R = \widetilde{E}(5.5) &= (3\angle 0°)\cdot e^{-(0.01133+j3.173)\cdot(5.5)} + (0.356)\cdot(3\angle 0°)\cdot e^{-2\cdot(0.01133+j3.173)\cdot(5.5)}\cdot e^{+(0.01133+j3.173)\cdot(5.5)} \\
&= (3\angle 0°)\cdot(0.1615 + j0.9256) + (0.356)\cdot(3\angle 0°)\cdot(-0.8307 + j0.2990)\cdot(0.1829 - j1.0485) \\
&= (0.4845 + j2.777) + (0.1725 + j0.9885) \\
&= (0.6570 + j3.765) = (3.822\angle 1.398\,\text{rad}) = \boxed{(3.822\angle 80.10°)\ \text{volts}}
\end{aligned}$$

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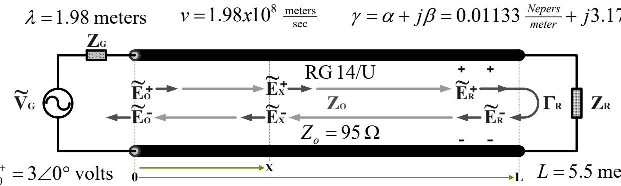
Example Problem

$\lambda = 1.98$ meters $\quad v = 1.98x10^8 \ \frac{\text{meters}}{\text{sec}} \quad \gamma = \alpha + j\beta = 0.01133 \ \frac{Nepers}{meter} + j3.173 \ \frac{\text{radians}}{\text{meter}}$

$\widetilde{V}_G = 6\angle 0°$ volts
$Z_G = 95\,\Omega$
$f = 100$ MHz



$\widetilde{E}_0^+ = 3\angle 0°$ volts

$Z_R = 200\,\Omega$
$\Gamma_R = 0.356$
$L = 5.5$ meters

Additionally, the **input impedance** of the line, $\boldsymbol{Z_{in}}$, is:

$$Z_{in} = Z_o \cdot \frac{1 + \Gamma_R \cdot e^{-2\cdot\gamma\cdot L}}{1 - \Gamma_R \cdot e^{-2\cdot\gamma\cdot L}} = (95)\cdot\frac{1 + (0.356)\cdot e^{-2\cdot(0.01133+j3.173)\cdot(5.5)}}{1 - (0.356)\cdot e^{-2\cdot(0.01133+j3.173)\cdot(5.5)}}$$

$$= (95)\cdot\frac{1 + (0.356)\cdot(-0.8307 + j0.2990)}{1 - (0.356)\cdot(-0.8307 + j0.2990)}$$

$$= \boxed{50.65 + j12.0\,\Omega}$$

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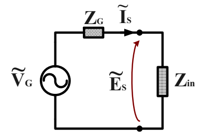
Example Problem

$\tilde{V}_G = 6\angle 0°$ volts
$Z_G = 95\,\Omega$
$f = 100$ MHz



$Z_{in} = 50.65 + j12.0\,\Omega$

Also note that, if the **input impedance** of the line is utilized, then the **steady-state current** can be solved using Ohm's Law and the **steady-state voltage** can be solved using a voltage-divider:

$$\tilde{I}_S = \frac{\tilde{V}_G}{Z_G + Z_{in}} = \frac{6\angle 0°}{95 + (50.65 + j12.0)} = (0.041\angle -4.70°)\text{ amps}$$

$$\tilde{E}_S = \tilde{V}_G \cdot \frac{Z_{in}}{Z_G + Z_{in}} = (6\angle 0°) \cdot \frac{50.65 + j12.0}{95 + (50.65 + j12.0)} = (2.14\angle 8.6°)\text{ volts}$$

*** - the same as previously solved**

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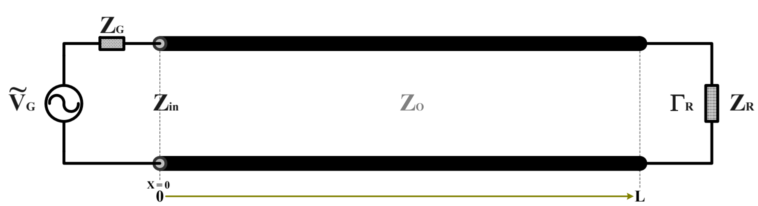
Impedance Characteristics on a Lossless Line



For a **lossless line**, the **propagation constant** is **purely imaginary**:

$$\gamma = \alpha + j\beta = 0 + j\beta = j\beta$$

Based on this, the **input impedance** expression can be rewritten as:

$$Z_{in} = Z_o \cdot \frac{1 + \Gamma_R \cdot e^{-j2\cdot\beta\cdot L}}{1 - \Gamma_R \cdot e^{-j2\cdot\beta\cdot L}} = Z_o \cdot \frac{1 + (\Gamma_R\angle -2\beta L)}{1 - (\Gamma_R\angle -2\beta L)}$$

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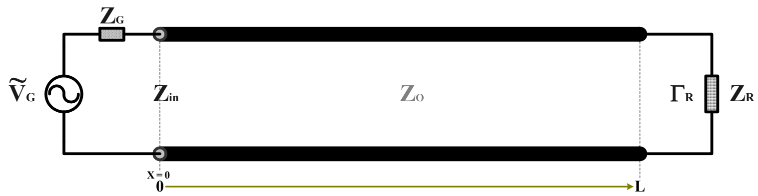
Impedance Characteristics on a Lossless Line



Since the value of a complex number expressed in polar form repeats with every 2π radian (or 360°) decrease in angle, the value of the **input impedance** of a lossless line:

$$Z_{in} = Z_o \cdot \frac{1 + (\Gamma_R \angle -2\beta L)}{1 - (\Gamma_R \angle -2\beta L)}$$

must repeat periodically whenever an increase in length causes the angle $-2\beta L$ to decrease by 2π radians.

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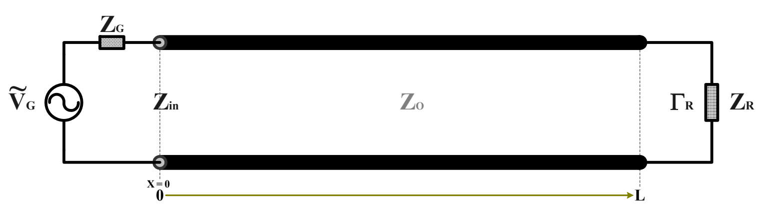
Impedance Characteristics on a Lossless Line



Thus, the **input impedance** of a **lossless line** will periodically repeat whenever the length of the line increases by **½ wavelength**:

$$-2\beta L = -2\pi \qquad \Rightarrow \qquad L = \frac{-2\pi}{-2\beta} = \frac{-2\pi}{-2 \cdot \frac{2\pi}{\lambda}} = \frac{\lambda}{2}$$

$$Z_{in} = Z_o \cdot \frac{1 + (\Gamma_R \angle -2\beta L)}{1 - (\Gamma_R \angle -2\beta L)}$$

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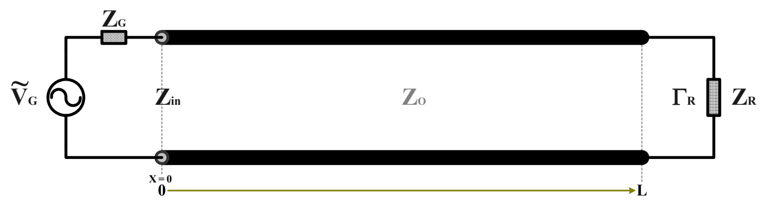
Impedance Characteristics on a Lossless Line



Furthermore, if the **length** of a lossless line is an **integer multiple** of **½ wavelength**, then the **input impedance** of the line will equal to the **load impedance** (independent of characteristic impedance):

$$Z_{in} = Z_o \cdot \frac{1+(\Gamma_R \angle -2\beta L)}{1-(\Gamma_R \angle -2\beta L)} = Z_o \cdot \frac{1+(\Gamma_R \angle -2\frac{2\pi}{\lambda}\frac{n\lambda}{2})}{1-(\Gamma_R \angle -2\frac{2\pi}{\lambda}\frac{n\lambda}{2})} = Z_o \cdot \frac{1+(\Gamma_R \angle -2\pi n)}{1-(\Gamma_R \angle -2\pi n)} = Z_o \cdot \frac{1+(\Gamma_R \angle 0)}{1-(\Gamma_R \angle 0)} = Z_R$$

$$Z_{in} = Z_R \quad \text{if} \quad L = n \cdot \frac{\lambda}{2}$$

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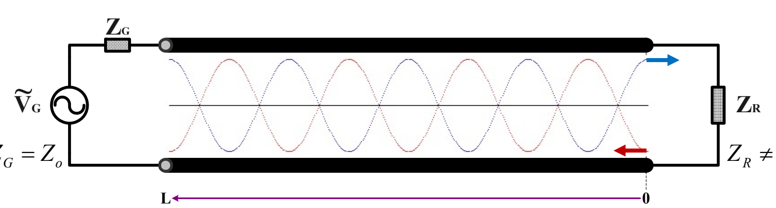
Standing Waves on a Transmission Line



$Z_G = Z_o$ $Z_R \neq Z_o$

Whenever a steady-state AC voltage is applied to an unmatched transmission-line, the resultant voltage on the line is equal to the instantaneous sum of the incident and reflected voltages:

$$\tilde{E}(x) = \tilde{E}_x^+ + \tilde{E}_x^-$$
$$= E_0^+ \cdot e^{-\gamma \cdot x} + \Gamma_R \cdot E_0^+ \cdot e^{-2\cdot\gamma\cdot L} \cdot e^{+\gamma\cdot x}$$

Note that the waveforms shown above result from a SC load ($\mathbf{Z_R = 0\Omega}$).

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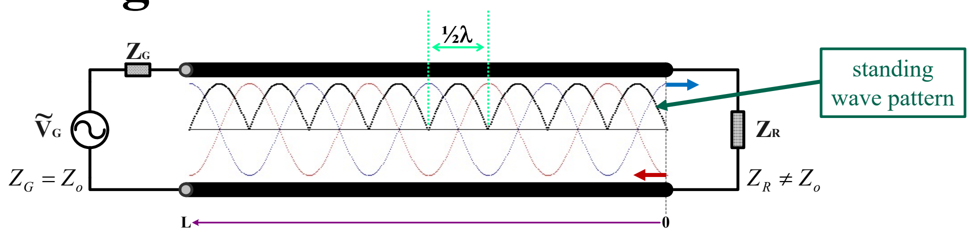
Standing Waves on a Transmission Line



It turns out that the **RMS magnitude** of the voltage at any position on the line is constant, the forming a **standing wave pattern**.

The **standing wave pattern** can be seen by plotting the RMS magnitude of the voltage as a function of position. Note that the pattern **repeats periodically** with every **½-wavelength** change in position on the line.

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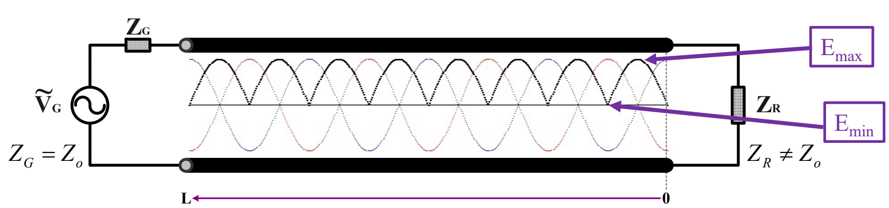
Voltage Standing Wave Ratio (VSWR)



The **Voltage Standing Wave Ratio** (**VSWR**) is defined as the ratio of the magnitude of the maximum RMS voltage in the standing-wave pattern over the magnitude of the adjacent minimum RMS voltage.

$$VSWR = \frac{|E_{\max}|}{|E_{\min}|}$$

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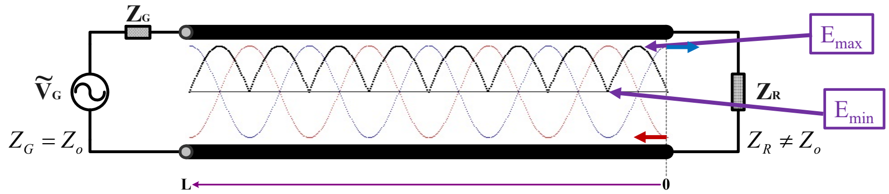
Voltage Standing Wave Ratio (VSWR)



The magnitude of the RMS voltages will be at a **maximum** whenever the incident and the reflected waves are **in-phase**, and at a **minimum** whenever the incident and the reflected waves are **out-of-phase by 180°**.

Thus:

$$|E_{\max}| = |E^+| + |E^-| \qquad\qquad |E_{\min}| = |E^+| - |E^-|$$

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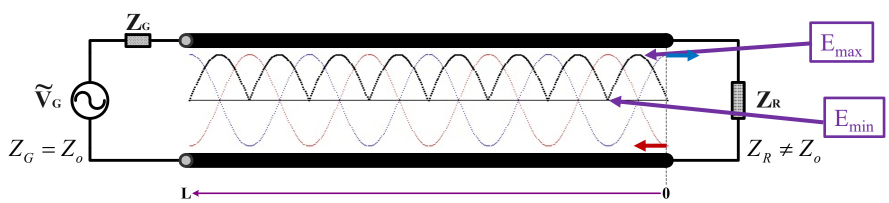
Voltage Standing Wave Ratio (VSWR)



The expressions for $|\boldsymbol{E}_{\mathbf{max}}|$ and $|\boldsymbol{E}_{\mathbf{min}}|$ can be substituted into the relationship for VSWR with the following result:

$$VSWR = \frac{|E_{\max}|}{|E_{\min}|} = \frac{|E^+| + |E^-|}{|E^+| - |E^-|} = \frac{1 + \frac{|E^-|}{|E^+|}}{1 - \frac{|E^-|}{|E^+|}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

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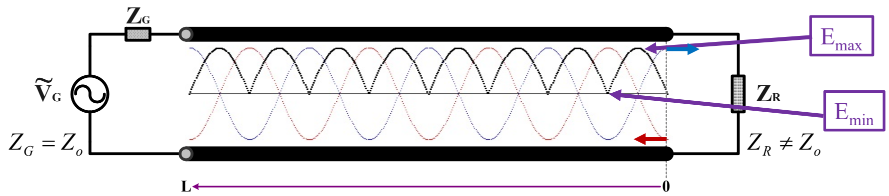
Voltage Standing Wave Ratio (VSWR)



Note that the following expressions:

$$VSWR = \frac{1+|\Gamma_R|}{1-|\Gamma_R|} \quad \leftrightarrows \quad |\Gamma_R| = \frac{VSWR-1}{VSWR+1}$$

holds true only if the line is assumed to be **lossless**, such that the magnitudes of the incident and reflected waves are constant.