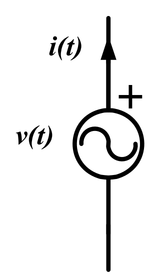
ECET 3000

Electrical Principles

Complex Power

AC Power – General Case

As previously stated, the **general expression** for the **power** produced by an AC source is:



$$p(t) = v(t) \cdot i(t)$$

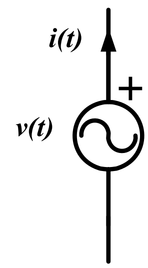
$$= V_{peak} \cdot I_{peak} \cdot \sin(\omega \cdot t + \phi) \cdot \sin(\omega \cdot t + \delta)$$

where: $v(t) = V_{peak} \cdot \sin(\omega \cdot t + \phi)$

$i(t) = I_{peak} \cdot \sin(\omega \cdot t + \delta)$

AC Power – General Case

If the voltages & currents are expressed in terms of their **RMS magnitudes**, the power expression becomes:



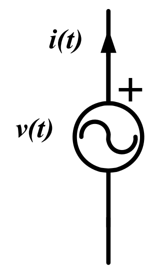
$$p(t) = v(t) \cdot i(t)$$
$$= \sqrt{2} \cdot V \cdot \sqrt{2} \cdot I \cdot \sin(\omega \cdot t + \phi) \cdot \sin(\omega \cdot t + \delta)$$
$$= 2 \cdot V \cdot I \cdot \sin(\omega \cdot t + \phi) \cdot \sin(\omega \cdot t + \delta)$$

which may be modified using several trigonometric identities into the following form:

$$p(t) = V \cdot I \cdot \cos(\phi - \delta)$$
$$- V \cdot I \cdot \cos(\phi - \delta) \cdot \cos(2 \cdot \omega \cdot t)$$
$$+ V \cdot I \cdot \sin(\phi - \delta) \cdot \sin(2 \cdot \omega \cdot t)$$

AC Power – General Case

The modified power expression is often simplified by defining a new variable, $\boldsymbol{\theta}$, where:



$$\theta = \phi - \delta$$

and substituting it into the equation, resulting in the **general power expression**:

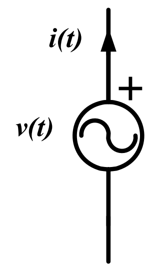
$$p(t) = V \cdot I \cdot \cos(\theta)$$
$$- V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t)$$
$$+ V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t)$$

AC Power – General Case

The angle $\boldsymbol{\theta}$:

$$\theta = \phi - \delta$$

which is defined by the difference between the phase angles of the voltage and current,



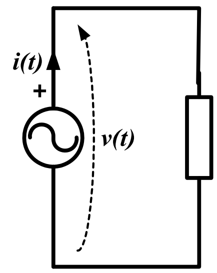
$$v(t) = V_{peak} \cdot \sin(\omega \cdot t + \phi)$$

$$i(t) = I_{peak} \cdot \sin(\omega \cdot t + \delta)$$

is often referred to as the **power angle**:

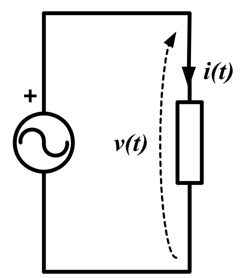
AC Power and Sources

This general expression defines the instantaneous **power produced by an AC source**.



$$p(t) = V \cdot I \cdot \cos(\theta) - V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t) + V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t)$$

Likewise, if the source is connected across a load that may have resistive, capacitive, and/or inductive components, then the solution also defines the instantaneous **power consumed by the AC supplied load**.



AC Power and Resistors

If an AC source is connected to a resistive load,

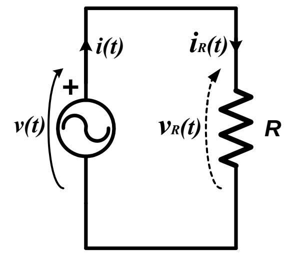
$$v_R(t) = \sqrt{2} \cdot V_R \cdot \sin(\omega \cdot t + \phi)$$

then the resistor current will be:

$$i_R(t) = \sqrt{2} \cdot \frac{V_R}{R} \cdot \sin(\omega \cdot t + \phi)$$

and the power consumed by the resistor will be:

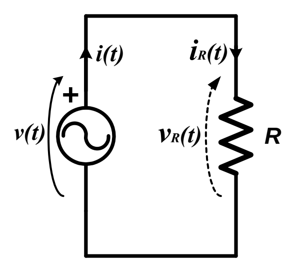
$$p_R(t) = V_R \cdot I_R - V_R \cdot I_R \cdot \cos(2 \cdot \omega \cdot t)$$



AC Power and Resistors

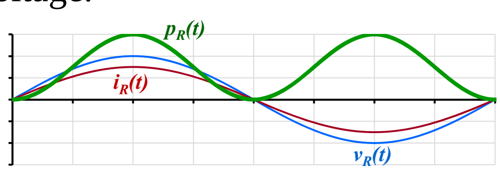
The resultant power waveform has two terms:

$$p_R(t) = \boxed{V_R \cdot I_R} - \boxed{V_R \cdot I_R \cdot \cos(2 \cdot \omega \cdot t)}$$



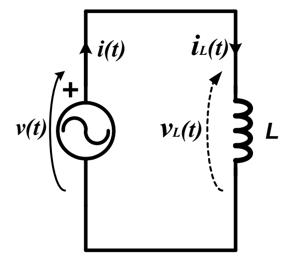
- the first of which is a **constant** that provides the **average** power supplied to the resistor, which is defined to be **Real Power**, P_R , and
- the second of which is a **purely sinusoidal** term that has a **zero average** value and varies at 2x the frequency of the source voltage.

$$P_R = V_R \cdot I_R \quad \text{Watts}$$



AC Power and Inductors

If the AC source is connected to an inductive load,



$$v_L(t) = \sqrt{2} \cdot V_L \cdot \sin(\omega \cdot t + \phi)$$

then the inductor current will be:

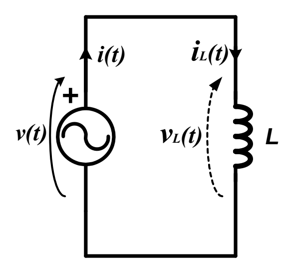
$$i_L(t) = \sqrt{2} \cdot \frac{V_L}{\omega \cdot L} \cdot \sin(\omega \cdot t + \phi° - 90°)$$

and the power consumed by the inductor will be:

$$p_L(t) = V_L \cdot I_L \cdot \sin(2 \cdot \omega \cdot t)$$

AC Power and Inductors

The resultant power waveform has only one term:



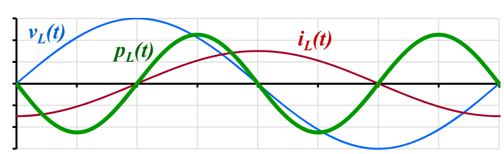
$$p_L(t) = \boxed{V_L \cdot I_L \cdot \sin(2 \cdot \omega \cdot t)}$$

which is a **purely sinusoidal** term that has a **zero average** (**real power**) value and varies at twice (2x) the frequency of the source voltage.

Although the inductor consumes zero **real power**:

$$P_L = 0 \quad \text{Watts}$$

power <u>is</u> instantaneously flowing into and out of the inductor.



AC Power and Capacitors

If the AC source is connected to an capacitor load,

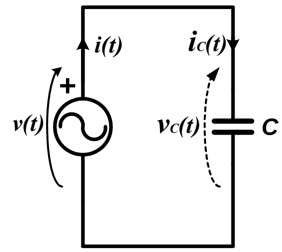
$$v_C(t) = \sqrt{2} \cdot V_C \cdot \sin(\omega \cdot t + \phi)$$

then the capacitor current will be:

$$i_C(t) = \sqrt{2} \cdot V_C \cdot \omega \cdot C \cdot \sin(\omega \cdot t + \phi° + 90°)$$

and the power consumed by the capacitor will be:

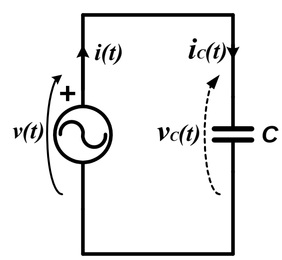
$$p_C(t) = -V_C \cdot I_C \cdot \sin(2 \cdot \omega \cdot t)$$



AC Power and Capacitors

The resultant power waveform has only one term:

$$p_C(t) = \boxed{-V_C \cdot I_C \cdot \sin(2 \cdot \omega \cdot t)}$$

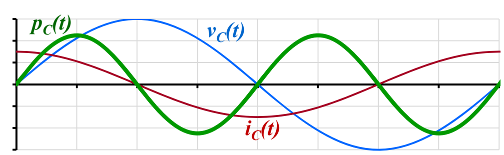


which is a **purely sinusoidal** term that has a **zero average** (**real power**) value and varies at twice (2x) the frequency of the source voltage.

Although the capacitor consumes zero **real power**:

$$P_C = 0 \quad \text{Watts}$$

power is instantaneously flowing into and out of the capacitor.



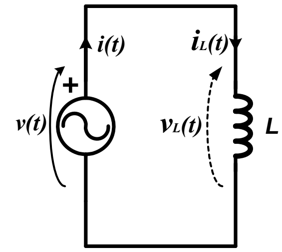
AC Power and Reactive Loads

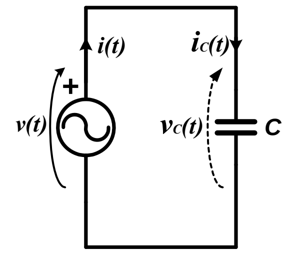
Although the (average) **real power consumed by both inductors and capacitors is zero**, there is power flowing in and out of these elements when supplied by an AC source:

$$p_L(t) = V_L \cdot I_L \cdot \sin(2 \cdot \omega \cdot t)$$

$$p_C(t) = -V_C \cdot I_C \cdot \sin(2 \cdot \omega \cdot t)$$

The term **Reactive Power** is used to characterize the amount of energy that is being temporarily stored and then released by a "***reactive load***" (capacitive or inductive).





Reactive Power

Reactive Power (***Q***) is defined as the magnitude of the power that is flowing into and out of a reactive load when supplied by an AC source.

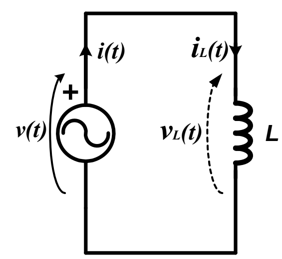
Thus, given: $p_L(t) = V_L \cdot I_L \cdot \sin(2 \cdot \omega \cdot t)$

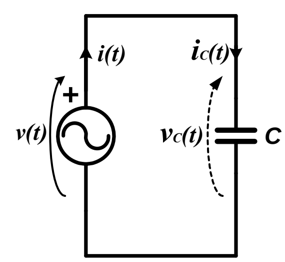
$$p_C(t) = -V_C \cdot I_C \cdot \sin(2 \cdot \omega \cdot t)$$

the reactive power for the inductive and capacitive loads can be defined as:

$$Q_L = +V_L \cdot I_L \quad \text{Vars}$$

$$Q_C = -V_C \cdot I_C \quad \text{Vars}$$





Reactive Power

Reactive Power is given the unit of "***Vars***", which stands for:

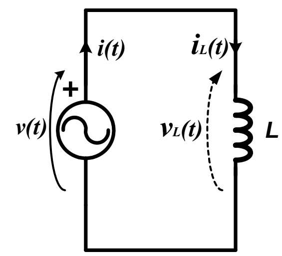
"***Volt-Amps-Reactive***".

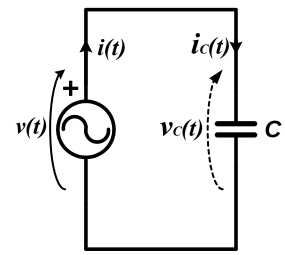
$$Q_L = +V_L \cdot I_L \quad \text{Vars}$$

$$Q_C = -V_C \cdot I_C \quad \text{Vars}$$

Note that the reactive power for an inductor is **positive** while the reactive power for a capacitor is **negative**.

Thus, it is often stated that an inductor "consumes" reactive power while a capacitor "produces" reactive power.





AC Power – General Case

If an AC source is connected to a complex impedance:

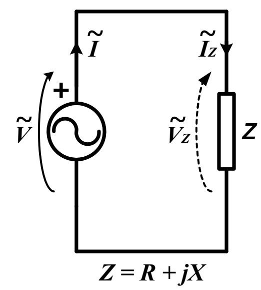
$$v(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

the resultant current can be expressed as:

$$i(t) = \sqrt{2} \cdot I \cdot \sin(\omega \cdot t + \delta)$$

and the power waveform as:

$$p(t) = v(t) \cdot i(t) = 2 \cdot V \cdot I \cdot \sin(\omega \cdot t + \phi) \cdot \sin(\omega \cdot t + \delta)$$



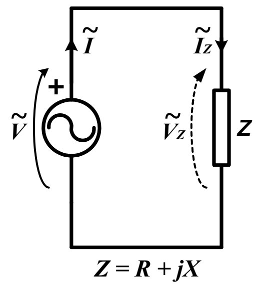
AC Power – General Case

The power expression:

$$p(t) = 2 \cdot V \cdot I \cdot \sin(\omega \cdot t + \phi) \cdot \sin(\omega \cdot t + \delta)$$

may be modified using several trigonometric identities into the following form:

$$\begin{aligned} p(t) = {} & V \cdot I \cdot \cos(\phi - \delta) \\ & - V \cdot I \cdot \cos(\phi - \delta) \cdot \cos(2 \cdot \omega \cdot t) \\ & + V \cdot I \cdot \sin(\phi - \delta) \cdot \sin(2 \cdot \omega \cdot t) \end{aligned}$$



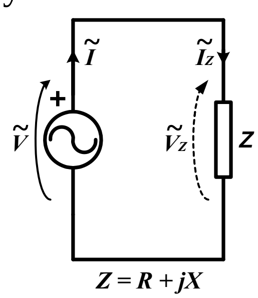
AC Power – General Case

The modified power expression is often simplified by defining a new variable, ***θ***, often referred to as the "power angle", where:

$$\theta = \angle\tilde{V} - \angle\tilde{I} = \phi - \delta$$

and substituting the variable into the equation:

$$\begin{aligned} p(t) = {} & V \cdot I \cdot \cos(\phi - \delta) \\ & - V \cdot I \cdot \cos(\phi - \delta) \cdot \cos(2 \cdot \omega \cdot t) \\ & + V \cdot I \cdot \sin(\phi - \delta) \cdot \sin(2 \cdot \omega \cdot t) \end{aligned}$$

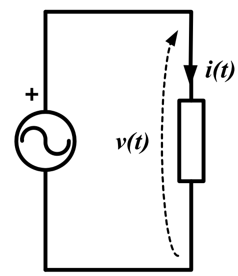


Note that the angle ***θ*** is also equal to the angle of the impedance **Z** when expressed in polar form.

$$Z = R + jX = |Z| \angle \theta$$

AC Power – General Case

The solution also provides both the instantaneous power produced by a source and the instantaneous power consumed by a load.



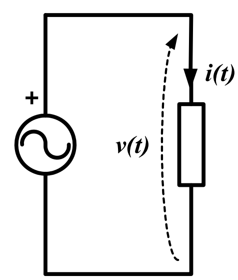
$$p(t) = \boxed{V \cdot I \cdot \cos(\theta)} - V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t) + V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t)$$

The first term is a **constant** that provides the average power or **Real Power** that is consumed by the resistive portion of the load:

$$P = V \cdot I \cdot \cos(\theta) \quad \text{Watts}$$

AC Power – General Case

The solution also provides both the instantaneous power produced by a source and the instantaneous power consumed by a load.



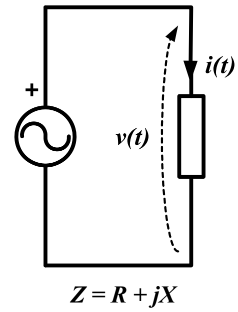
$$p(t) = V \cdot I \cdot \cos(\theta) \boxed{- V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t)} + V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t)$$

The second term is **purely sinusoidal** and provides the time-varying portion of the power that is consumed by the resistive portion of the load.

AC Power – General Case

The solution also provides both the instantaneous power produced by a source and the power consumed by a load.

$$p(t) = V \cdot I \cdot \cos(\theta) - V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t) + V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t)$$

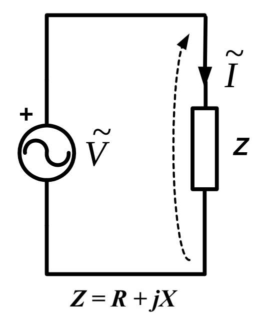


The third term is also purely sinusoidal, the magnitude of which provides the **Reactive Power** consumed by the reactive portion of the load.

$$Q = V \cdot I \cdot \sin(\theta) \quad \text{Vars}$$

Complex Power

The term **Complex Power** is used to characterize both the Real Power and the Reactive Power in an AC system that is being supplied to a complex load impedance (that may have a resistive and/or a reactive component).



$$Z = R + jX$$

Complex Power (***S***) is a complex number and is defined by:

$$S = P + jQ$$

where: ***P*** is Real Power, and

Q is Reactive Power.

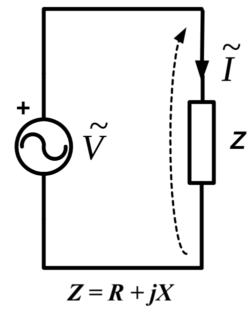
Complex Power

Complex Power (*S*):

$$S = P + jQ$$

may be solved directly from the applied phasor voltage and the resultant phasor current as:

$$\boxed{S = \tilde{V} \cdot \tilde{I}^*}$$
$$= (V\angle\phi)\cdot(I\angle-\delta)$$
$$= V\cdot I\angle(\phi-\delta) = V\cdot I\angle\theta$$
$$= V\cdot I\cdot\cos\theta + jV\cdot I\cdot\sin\theta$$
$$= P + jQ$$

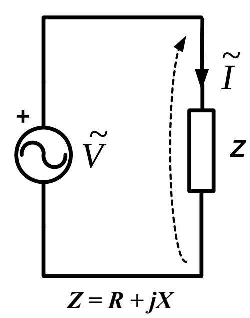


Complex Power

Note that $\tilde{I}^*$ is the <u>complex conjugate</u> of $\tilde{I}$ and is defined as:

$$\tilde{I}^* = (I\angle\delta)^* = (I\angle-\delta)$$

In other words, the **complex conjugate** of a complex number in polar form has the <u>same magnitude</u> as the original number but its <u>angle is negated</u>.



For example, given the current:

$$\tilde{I} = 24\angle30° \text{ amps}$$

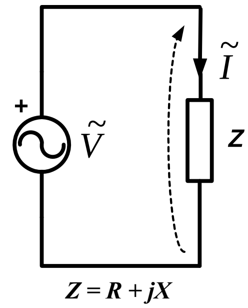
The **complex conjugate** of the current, $\tilde{I}^*$, is:

$$\tilde{I}^* = (I\angle\delta)^* = (I\angle-\delta) = 24\angle-30° \text{ amps}$$

Apparent Power

Apparent Power ($|S|$) is defined to be the magnitude of complex power:

$$|S| = V \cdot I = \sqrt{P^2 + Q^2} \quad \text{VA}$$



Apparent power is often specified as part of the ratings of an electric machine, such that:

$$|S|_{rated} = V_{rated} \cdot I_{rated} \quad \text{VA}$$

Note that, given two of these rated values (voltage, current, apparent power), the third can be calculated from the above equation.

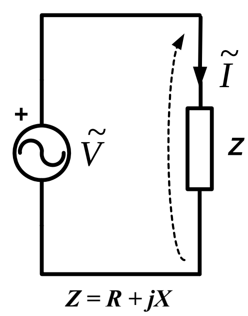
For example, given a **500VA** transformer whose primary winding is rated at **208V**, the **rated current** for the primary winding is:

$$I_{rated} = \frac{|S|_{rated}}{V_{rated}} = \frac{500\,\text{VA}}{208\,\text{V}} = 2.40\ \text{A}$$

Power Factor

Power Factor (pf) is defined as the ratio of the real power over the apparent power:

$$pf = \frac{P}{|S|}$$



Thus, power factor may be solved as:

$$pf = \frac{P}{|S|} = \frac{V \cdot I \cdot \cos\theta}{V \cdot I} = \cos\theta$$

Power Factor is often characterized by a qualifier, either **leading** or **lagging**, which is used to describe the phase angle relationship between the applied voltage and the resultant current waveforms.

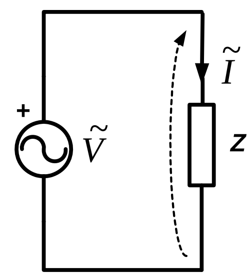
Power Factor

A **leading** power factor exists when the current "leads" the voltage, which occurs when the load impedance has an overall **capacitive** aspect, resulting in a negative angle $\boldsymbol{\theta}$:

$$Z = R - jX_C \qquad -90° \le \theta < 0°$$

A **lagging** power factor exists when the current "lags" the voltage, which occurs when the load impedance has an overall **inductive** aspect, resulting in a positive angle difference for $\boldsymbol{\theta}$:

$$Z = R + jX_L \qquad 0° < \theta \le +90°$$



$$\tilde{V} = V\angle\phi°$$
$$\tilde{I} = I\angle\delta°$$
$$Z = R + jX = |Z|\angle\theta$$
$$\theta = \phi - \delta$$

Power Factor

Note that a **purely resistive load** results in a zero value for the angle $\boldsymbol{\theta}$, which is neither leading nor lagging.

$$Z = R \qquad \theta = 0°$$

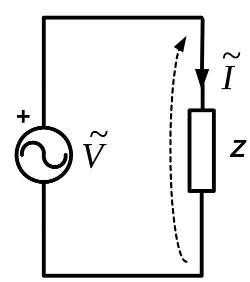
This is often referred to as a **unity power factor** since the value of power factor under this condition equals one.

$$\cos(\theta) = \cos(0°) = 1$$

Note that, in the case of a **purely resistive load**, **rated <u>real power</u>** may be specified in place of rated apparent power since, for a **purely resistive load**:

$$\theta = 0° \quad \Rightarrow \quad P = V \cdot I \cdot \cos\theta = V \cdot I \cdot \cos 0° = V \cdot I$$

$$\therefore \quad P_{rated} = V_{rated} \cdot I_{rated} \qquad \text{(purely resistive loads only)}$$



$$\tilde{V} = V\angle\phi°$$
$$\tilde{I} = I\angle\delta°$$
$$Z = R + jX = |Z|\angle\theta$$
$$\theta = \phi - \delta$$

Summary of Complex Power Equations

Complex Power (S): $S = \tilde{V} \cdot \tilde{I}^* = P + jQ$

Real Power (P): $P = V \cdot I \cdot \cos\theta$

Reactive Power (Q): $Q = V \cdot I \cdot \sin\theta$

Be careful using these equations

Apparent Power ($|S|$): $|S| = V \cdot I = \sqrt{P^2 + Q^2}$

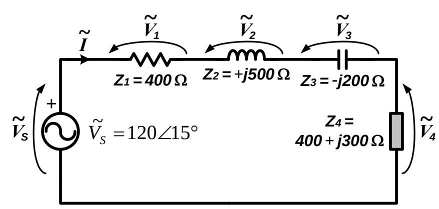
Power Factor (pf): $pf = \cos\theta$

The formulas shown for **Real Power** and **Reactive Power** are often mistakenly applied. For that reason, I highly recommend calculating **Complex Power** whenever either Real or Reactive Power is required, and then:
setting **Real Power** equal to the **real portion** of the **Complex Power**, and
setting **Reactive Power** equal to the **imaginary portion** of the **Complex Power**.

$S = P + jQ \quad \therefore \quad P = Real\{S\} \quad Q = Imag\{S\}$

Complex Power

Determine the **complex power** produced by the **source** in the following circuit:



$$Z_{eq} = Z_1 + Z_2 + Z_3 + Z_4$$
$$= (400) + (+j500) + (-j200) + (400 + j300)$$
$$= (800 + j600)\,\Omega$$

$$\tilde{I} = \frac{\tilde{V}_S}{Z_{eq}} = \frac{120\angle 15^\circ}{800 + j600} = 0.12\angle -21.87^\circ \text{ amps}$$

The **apparent power** produced by the source is:
$|S_S| = V_S \cdot I = (120)\cdot(0.12)$
$= 14.4$ VA

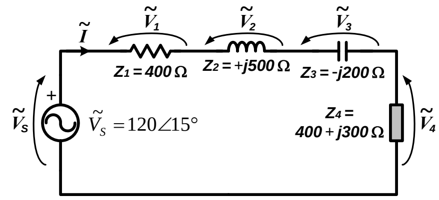
$$S_{Source} = \tilde{V}_S \cdot \tilde{I}^* = (120\angle 15^\circ)\cdot(0.12\angle +21.87) = 11.52 + j8.64$$

$P_{Source} = 11.52$ watts

$Q_{Source} = 8.64$ VARs

Complex Power

Determine the **complex power** consumed by **loads *Z_1* and *Z_2*.**



$$\tilde{V}_1 = \tilde{I} \cdot Z_1 = (0.12\angle -21.87°)\cdot(400)$$
$$= 48\angle -21.87° \text{ volts}$$

$$\boxed{S_1} = \tilde{V}_1 \cdot \tilde{I}^*$$
$$= (48\angle -21.87°)\cdot(0.12\angle +21.87)$$
$$= \boxed{5.76 + j0}$$

$\boxed{P_1 = 5.76 \text{ watts}}$ $\boxed{Q_1 = 0 \text{ VARs}}$

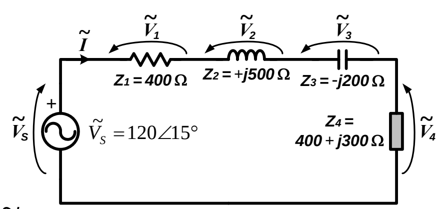
$$\tilde{V}_2 = \tilde{I} \cdot Z_2 = (0.12\angle -21.87°)\cdot(+j500)$$
$$= 60\angle 68.13° \text{ volts}$$

$$\boxed{S_2} = \tilde{V}_2 \cdot \tilde{I}^*$$
$$= (60\angle 68.13°)\cdot(0.12\angle +21.87)$$
$$= \boxed{0 + j7.2}$$

$\boxed{P_2 = 0 \text{ watts}}$ $\boxed{Q_2 = 7.2 \text{ VARs}}$

Complex Power

Determine the **complex power** consumed by **loads *Z_3* and *Z_4*.**



$$\tilde{V}_3 = \tilde{I} \cdot Z_3 = (0.12\angle -21.87°)\cdot(-j200)$$
$$= 24\angle -111.87° \text{ volts}$$

$$\boxed{S_3} = \tilde{V}_3 \cdot \tilde{I}^*$$
$$= (24\angle -111.87°)\cdot(0.12\angle +21.87)$$
$$= \boxed{0 - j2.88}$$

$\boxed{P_3 = 0 \text{ watts}}$ $\boxed{Q_3 = 2.88 \text{ VARs}}$

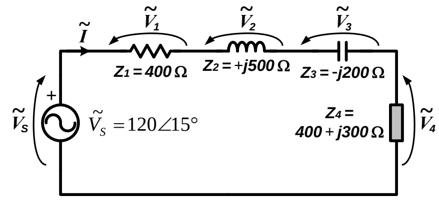
$$\tilde{V}_4 = \tilde{I} \cdot Z_4$$
$$= (0.12\angle -21.87°)\cdot(400 + j300)$$
$$= 60\angle 15° \text{ volts}$$

$$\boxed{S_4} = \tilde{V}_4 \cdot \tilde{I}^*$$
$$= (60\angle 15°)\cdot(0.12\angle +21.87)$$
$$= \boxed{5.76 + j4.32}$$

$\boxed{P_4 = 5.76 \text{ watts}}$ $\boxed{Q_4 = 4.32 \text{ VARs}}$

Complex Power

Compare the **complex power** produced by the **source** and the **complex powers** consumed by the **loads**.



$$S_{Source} = 11.52 + j8.64$$

$$S_1 = 5.76 + j0 \quad S_2 = 0 + j7.2 \quad S_3 = 0 - j2.88 \quad S_4 = 5.76 + j4.32$$

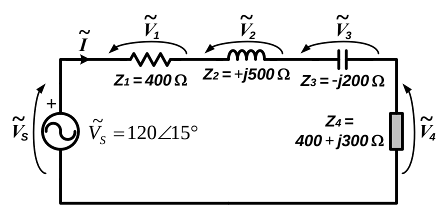
$$\begin{aligned} S_{Total(Loads)} &= S_1 + S_2 + S_3 + S_4 \\ &= (5.76 + j0) + (0 + j7.2) + (0 - j2.88) + (5.76 + j4.32) \\ &= 11.52 + j8.64 \end{aligned}$$

Note that: $P_{Total(Loads)} = P_1 + P_2 + P_3 + P_4 = P_{Source}$

$Q_{Total(Loads)} = Q_1 + Q_2 + Q_3 + Q_4 = Q_{Source}$

Complex Power

Determine the **power factor** of the **source** when supplying the four loads.



$$\tilde{V}_S = 120\angle 15° \text{ volts} = V\angle \phi°$$

$$\tilde{I} = 0.12\angle -21.87° \text{ amps} = I\angle \delta°$$

$$\theta_S = \angle\tilde{V}_S - \angle\tilde{I} = \phi° - \delta° = 15° - (-21.87°) = 36.87°$$

$$pf = \cos\theta_S = \cos(36.87°) = 0.8 \text{ lagging}$$

The power factor is **lagging** because the overall equivalent load is resistive and inductive, resulting in a positive angle θ_S.
$0° < \theta \leq +90°$

Note that the **power angle** θ_S calculated for the source:
$\theta_S = \angle\tilde{V}_S - \angle\tilde{I} = \phi° - \delta° = 15° - (-21.87°) = 36.87°$
is equal to the angle of the overall equivalent load when expressed in polar form:
$Z_{eq} = |Z_{eq}|\angle\theta_{eq} = (800 + j600) = (1000\angle 36.87°)\,\Omega$