ECET 3000

Electrical Principles

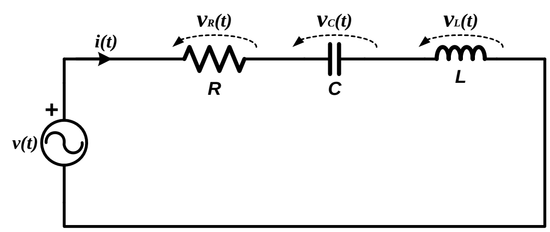
Analysis of AC Circuits

Analysis of AC Sourced R-L-C Circuits

For example: Given a simple, series-connected, R-L-C, AC circuit, a **2nd-order differential equation** must be solved in order to determine the source current.

But, provided that the source is sinusoidally-varying, a **Phasor Analysis** can be performed on the circuit, the method of which allows for the *V-I* relationships for all three elements to be linearized into simple "Ohm's Law" based relationships.

$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L} \cdot \frac{di(t)}{dt} + \frac{1}{LC} i(t) = \frac{1}{L} \cdot \frac{dv(t)}{dt}$$



Phasor Representation of Sine Waves

A **phasor** is a representation of a sine-wave whose magnitude, phase and frequency are constant.

Phasors reduce the dependency of these parameters to three independent factors, thus allowing for the simplification of certain types of calculations.

It turns out that, for steady-state AC circuits, the time dependency of the sine-waves can be factored out, reducing the non-linear differential equations required for their solution to a simpler set of linear, algebraic equations.

Phasors and AC Voltages

The sinusoidal voltage:

$$v(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

may be defined in the form of a ***phasor voltage***:

$$\tilde{V} = Ve^{j\phi} = V\angle\phi$$

in which the voltage is expressed as a complex number in "***polar***" form, having the RMS magnitude V and the phase angle ϕ.

(Note – although the phasor value may be expressed in terms of "peak" magnitudes, RMS voltage magnitudes will be utilized in this course unless specifically stated otherwise.)

Phasors and AC Voltages

For example, given the sinusoidal voltage:

$$v(t) = 100 \cdot \sin(377 \cdot t + 30°) \quad \text{volts}$$

which may be expressed in terms of its RMS magnitude:

$$v(t) = \sqrt{2} \cdot 70.7 \cdot \sin(377 \cdot t + 30°) \quad \text{volts}$$

The **phasor representation** of this voltage is:

$$v(t) \quad \Leftrightarrow \quad \tilde{V} = 70.7 e^{j\frac{\pi}{6}} = 70.7\angle 30° \quad \text{volts}$$

such that: $30° = 30° \cdot \dfrac{2\pi \text{ radians}}{360°} = \dfrac{\pi}{6} \text{ radians}$

Phasors and AC Currents

The sinusoidal current:

$$i(t) = \sqrt{2} \cdot I \cdot \sin(\omega \cdot t + \delta)$$

may also be defined in the form of a ***phasor current***:

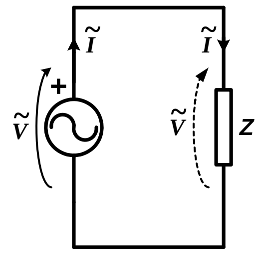
$$\tilde{I} = Ie^{j\delta} = I\angle\delta$$

in which the current is expressed as a complex number in "***polar***" form, having the RMS magnitude I and the phase angle δ.

(Note –RMS current magnitudes will also be utilized in this course unless specifically stated otherwise.)

Impedance

The **<u>impedance</u>** of a load provides a measure of the response that the load will have when supplied by a steady-state AC waveform.



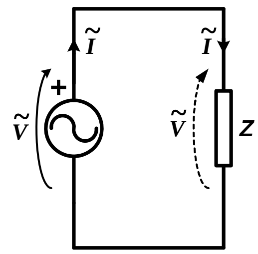
Specifically, the **impedance** value of a load, ***Z***, is defined as the ratio of the ***phasor voltage*** that is applied across the load over the ***phasor current*** that flows through the load:

$$Z = \frac{\tilde{V}}{\tilde{I}} \quad (\Omega)$$

Impedance

Based on the impedance expression:

$$Z = \frac{\tilde{V}}{\tilde{I}}$$



an Ohm's Law type of relationship between the phasor values of the load voltage and current can be defined:

$$\tilde{V} = \tilde{I} \cdot Z$$

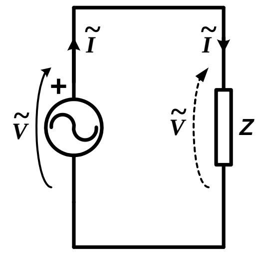
which means that any of the DC circuit theory that was derived based on Ohm's Law can also be applied to steady-state AC circuits whose loads are expressed as impedances and whose voltages and currents are expressed by their phasor values.

Impedance

Thus, given the phasor values of the load voltage and current:

$$\tilde{V} = V\angle\phi \qquad \tilde{I} = I\angle\delta$$

the ***impedance***, *Z*, can be expressed in terms of their phasor values as :



$$Z = |Z|\angle\theta = \frac{\tilde{V}}{\tilde{I}} = \frac{V\angle\phi}{I\angle\delta} = \frac{V}{I}\angle(\phi - \delta)$$

where: $|Z| = \frac{V}{I}$ $\theta = \phi - \delta$

Note – Impedance is typically expressed as a complex number written in "rectangular" form: *Z=R+jX*

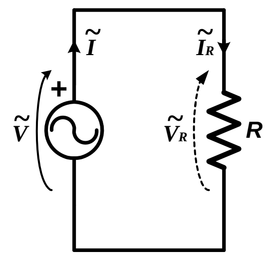
Impedance of a Resistor

Given the voltage across a resistor:

$$v_R(t) = \sqrt{2}\cdot V_R \cdot \sin(\omega\cdot t + \phi)$$

the current flowing through the resistor will be:

$$i_R(t) = \sqrt{2}\cdot \frac{V_R}{R} \cdot \sin(\omega\cdot t + \phi)$$



$$v_R(t) = i_R(t)\cdot R$$

When expressed as phasors, the resistor's voltage and current can be rewritten as:

$$\tilde{V}_R = V_R\angle\phi \qquad \tilde{I}_R = \frac{V_R}{R}\angle\phi$$

Impedance of a Resistor

Based on the values of its phasor voltage and current:

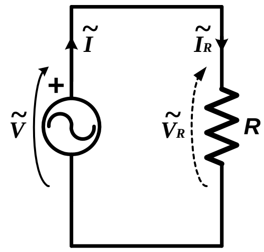
$$\tilde{V}_R = V_R \angle \phi \qquad \tilde{I}_R = \frac{V_R}{R} \angle \phi$$

the **impedance of the resistor** can be defined as:

$$Z_R = \frac{\tilde{V}_R}{\tilde{I}_R} = \frac{V_R \angle \phi}{\left(\frac{V_R}{R}\right) \angle \phi} = R \angle 0° = R + j0$$

Thus, the impedance of a resistor is equal to its resistance, which is a purely **real** value:

$$Z_R = R$$



$$v_R(t) = i_R(t) \cdot R$$

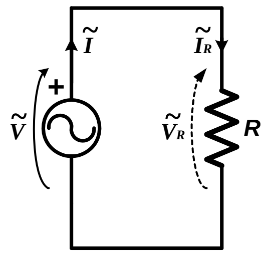
Impedance of a Resistor

Thus, not only does Ohm's Law hold true for resistors that are supplied with both DC voltages and time-varying (AC) voltages:

$$V_R = I_R \cdot R \qquad v_R(t) = i_R(t) \cdot R$$

Ohm's Law also holds true for resistors whose voltages and currents are expressed as phasors:

$$\tilde{V}_R = \tilde{I}_R \cdot Z_R = \tilde{I}_R \cdot R$$



$$v_R(t) = i_R(t) \cdot R$$

$$\tilde{V}_R = \tilde{I}_R \cdot R$$

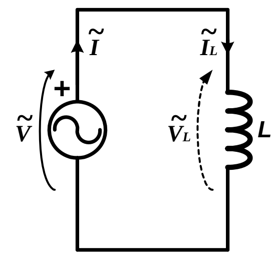
Impedance of an Inductor

Given the voltage across an inductor:

$$v_L(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

the current flowing through the inductor will be:

$$i_L(t) = \sqrt{2} \cdot \frac{V}{\omega \cdot L} \cdot \sin(\omega \cdot t + \phi^\circ - 90^\circ)$$



$$v_L(t) = L \cdot \frac{di_L(t)}{dt}$$

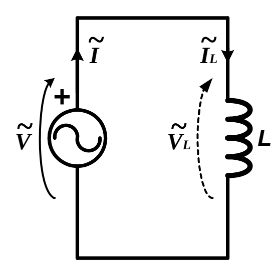
When expressed as phasors, the inductor's voltage and current can be rewritten as:

$$\tilde{V}_L = V_L \angle \phi \qquad \tilde{I}_L = \frac{V_L}{\omega \cdot L} \angle \phi - 90^\circ$$

Impedance of an Inductor

Based on the inductor's phasor voltage and current:

$$\tilde{V}_L = V_L \angle \phi \qquad \tilde{I}_L = \frac{V_L}{\omega \cdot L} \angle \phi - 90^\circ$$



$$v_L(t) = L \cdot \frac{di_L(t)}{dt}$$

the **impedance of the inductor** can be defined as:

$$Z_L = \frac{\tilde{V}_L}{\tilde{I}_L} = \frac{V_L \angle \phi}{\left(\frac{V_L}{\omega \cdot L} \right) \angle \phi - 90^\circ} = (\omega \cdot L) \angle + 90^\circ$$

which can be expressed in rectangular form as:

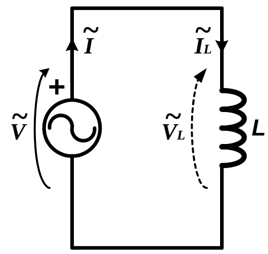
$$Z_L = (\omega \cdot L) \angle + 90^\circ = 0 + j\omega \cdot L = j\omega \cdot L$$

Impedance of an Inductor

Thus, based on its phasor voltage and current:

$$\tilde{V}_L = V_L \angle \phi \qquad \tilde{I}_L = \frac{V_L}{\omega \cdot L} \angle \phi - 90°$$

the **impedance of the inductor** can be defined as:



$$Z_L = (\omega \cdot L) \angle +90° = j\omega \cdot L$$

which is a **positive imaginary number**.

$$v_L(t) = L \cdot \frac{di_L(t)}{dt}$$

And, when expressed as an impedance, Ohm's Law holds true for an inductor whose voltage and current are expressed as phasors.

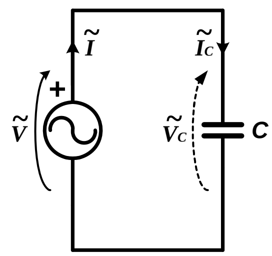
$$\tilde{V}_L = \tilde{I}_L \cdot Z_L$$

Impedance of an Capacitor

Given the voltage across a capacitor:

$$v_C(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

the current flowing through the capacitor will be:



$$i_C(t) = \sqrt{2} \cdot V \cdot \omega \cdot C \cdot \sin(\omega \cdot t + \phi° + 90°)$$

When expressed as phasors, the capacitor's voltage and current can be rewritten as:

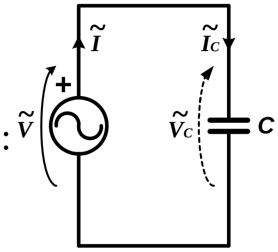
$$i_C(t) = C \cdot \frac{dv_C(t)}{dt}$$

$$\tilde{V}_C = V_C \angle \phi \qquad \tilde{I}_C = V_C \cdot \omega \cdot C \angle \phi + 90°$$

Impedance of an Capacitor

Based on the capacitor's phasor voltage and current:

$$\tilde{V}_C = V_C \angle \phi \qquad \tilde{I}_C = V_C \cdot \omega \cdot C \angle \phi + 90°$$



the **impedance of the capacitor** can be defined as:

$$Z_C = \frac{\tilde{V}_C}{\tilde{I}_C} = \frac{V \angle \phi}{V \cdot \omega \cdot C \angle \phi + 90°} = \frac{1}{\omega \cdot C} \angle -90°$$

$$i_C(t) = C \cdot \frac{dv_C(t)}{dt}$$

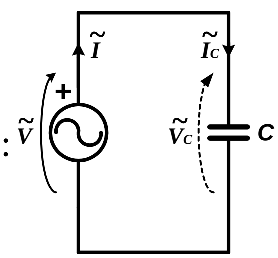
which can be expressed in rectangular form as:

$$Z_C = \frac{1}{\omega \cdot C} \angle -90° = 0 - j\frac{1}{\omega \cdot C} = -j\frac{1}{\omega \cdot C}$$

Impedance of an Capacitor

Thus, based on its phasor voltage and current:

$$\tilde{V}_C = V_C \angle \phi \qquad \tilde{I}_C = V_C \cdot \omega \cdot C \angle \phi + 90°$$



the **impedance of the capacitor** can be defined as:

$$Z_C = \frac{1}{\omega \cdot C} \angle -90° = -j\frac{1}{\omega \cdot C}$$

$$i_C(t) = C \cdot \frac{dv_C(t)}{dt}$$

which is a **negative imaginary number**.

And, when expressed as an impedance, Ohm's Law holds true for a capacitor whose voltage and current are expressed as phasors.

$$\tilde{V}_C = \tilde{I}_C \cdot Z_C$$

Reactance

Reactance defines the manner in which **inductive** and **capacitive** loads react to a steady-state sinusoidal voltage.

The reactance of an inductive or capacitive load is equal to the magnitude of the load's impedance value.

Therefore:

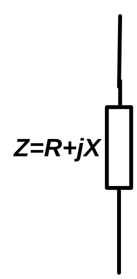
- the reactance of a resistor is: $X_R = 0\ \Omega$
- the reactance of an inductor is: $X_L = \omega \cdot L\ \Omega$
- the reactance of a capacitor is: $X_C = \dfrac{1}{\omega \cdot C}\ \Omega$

Complex Impedances

A complex impedance ***Z*** is an impedance can have both resistive and reactive (inductive or capacitive) components, and may be expressed in the form:

$$Z = R + jX$$

where: R is the resistive component of the load, and
X is the reactive component of the load.



Note:
- the impedance of a resistor is: $Z_R = R$
- the impedance of an inductor is: $Z_L = jX_L = j(\omega \cdot L)$
- the impedance of a capacitor is: $Z_C = -jX_C = -j\left(\dfrac{1}{\omega \cdot C}\right)$

Phasor Analysis of AC Circuits

If a voltage source having the phasor value:

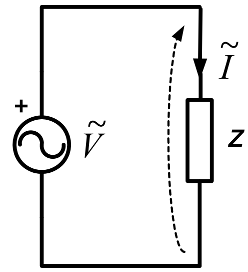
$$\tilde{V} = V\angle\phi$$

is applied across the complex impedance:

$$Z = R + jX$$

then the phasor value of the current may be solved by applying Ohm's Law:

$$\tilde{I} = \frac{\tilde{V}}{Z} = \frac{V\angle\phi}{R + jX} = I\angle\delta$$



The Range of Impedance Values

Given a complex impedance:

$$Z = R + jX$$

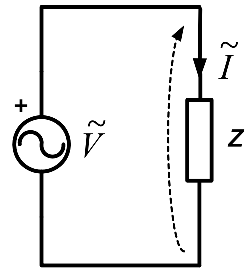
- The "real" portion of the impedance can be in the range:

$$0 \le R \le +\infty$$

- The "imaginary" portion of the impedance can be in the range:

$$-\infty \le X \le +\infty$$

if R and X relate only to passive loads.
(I.e. – resistors, inductors, and/or capacitors)



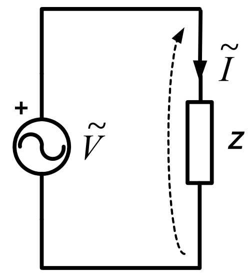
Voltage & Current Phase Angles

If the impedance is expressed in polar form:

$$Z = R + jX = |Z|\angle\theta°$$

then the angle θ will fall within the range:

$$-90° \le \theta \le +90°.$$



And, since:

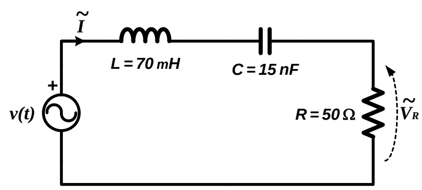
$$\tilde{I} = \frac{\tilde{V}}{Z} = \frac{V\angle\phi°}{|Z|\angle\theta°} = \frac{V}{|Z|}\angle\phi° - \theta° = I\angle\delta°$$

the phase angle of the current will be within ±90° of the phase angle of the voltage:

$$\phi° - 90° \le \delta° \le \phi° + 90°$$

Phasor Analysis Example

Perform a phasor analysis of the circuit shown below in order to determine the phasor values of the source current and the resistor voltage:



if:

$$v(t) = \sqrt{2} \cdot 24 \cdot \sin(\omega \cdot t + 30°) \text{ volts}$$

$$f = 60 \text{ Hz}$$

Phasor Analysis Example

Step 1 – Express the source voltage by its phasor value and all circuit elements as impedances.

$$v(t) = \sqrt{2}\cdot 24\cdot\sin(\omega\cdot t + 30°)\text{ volts}$$

$$f = 60\ Hz$$

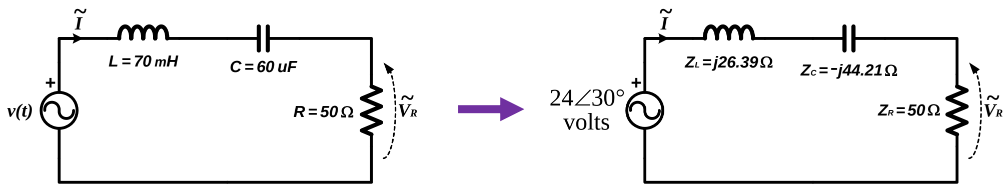
$$\omega = 2\cdot\pi\cdot f \approx 377\text{ rad/sec}$$

$$\tilde{V} = 24\angle 30° = 24e^{j\frac{\pi}{6}\text{rad}}\text{ volts}$$

$$Z_L = jX_L = j\omega\cdot L = j\left(377\cdot 70x10^{-3}\right) = +j26.39\ \Omega$$

$$Z_C = jX_C = -j\frac{1}{\omega\cdot C} = -j\frac{1}{377\cdot 60x10^{-6}} = -j44.21\,\Omega$$

$$Z_R = R = 50\ \Omega$$



Phasor Analysis Example

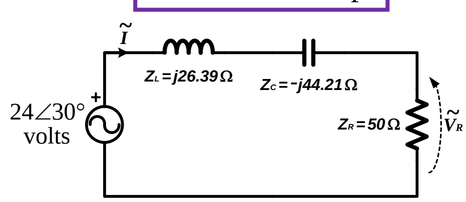
Step 2 – Solve for the phasor value of the source current.

$$\tilde{I} = \frac{\tilde{V}}{Z_{Total}} = \frac{\tilde{V}}{Z_L + Z_C + Z_R}$$

$$= \frac{24\angle 30°}{+j26.39 - j44.21 + 50}$$

$$= \frac{24\angle 30°}{50 - j17.82}$$

$$= \boxed{0.452\angle 49.6°\text{ amps}}$$



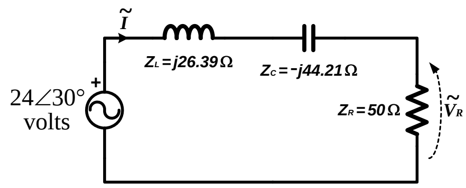
Phasor Analysis Example

Step 3 – Solve for the phasor value of the resistor voltage.

$$\tilde{V}_R = \tilde{V} \cdot \frac{Z_R}{Z_{Eq}} = \tilde{V} \cdot \frac{Z_R}{Z_L + Z_C + Z_R}$$

$$= 24\angle 30° \cdot \frac{50}{50 - j17.82}$$

$$= \boxed{22.6\angle 49.6° \text{ volts}}$$



Phasor Analysis Example

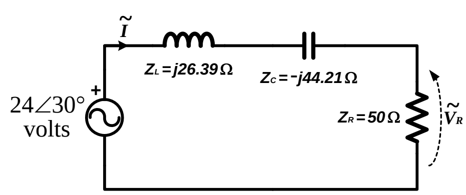
Step 4 – If needed, convert the phasor values into sinusoidal expressions.

$$\tilde{I} = 0.452\angle 49.6° \text{ amps}$$

$$i(t) = \sqrt{2} \cdot 0.452 \cdot \sin(377 \cdot t + 49.6°) \text{ amps}$$

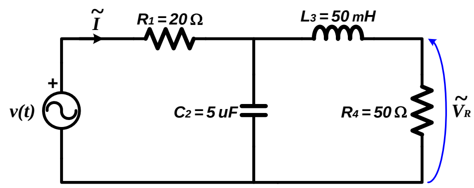
$$\tilde{V}_R = 22.6\angle 49.6° \text{ volts}$$

$$v_R(t) = \sqrt{2} \cdot 22.6 \cdot \sin(377 \cdot t + 49.6°) \text{ volts}$$



Phasor Analysis Example #2

Perform a phasor analysis of the circuit shown below in order to determine the phasor value of the voltage across resistor R_4:



if:

$$v(t) = \sqrt{2} \cdot 24 \cdot \sin(\omega \cdot t + 30°) \text{ volts}$$

$$\omega = 1000 \text{ rad/sec}$$

Phasor Analysis Example #2

Step 1 – Express the source voltage by its phasor value and all circuit elements as impedances.

$$v(t) = \sqrt{2} \cdot 24 \cdot \sin(\omega \cdot t + 30°) \text{ volts}$$

$$\omega = 1000 \text{ rad/sec}$$

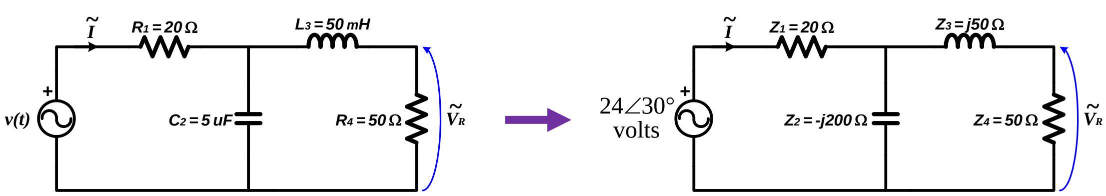
$$\tilde{V} = 24\angle 30° = 24e^{j\frac{\pi}{6}} \text{ volts}$$

$$Z_1 = 20\ \Omega$$

$$Z_2 = -j\left(\frac{1}{1000 \cdot 5x10^{-6}}\right) = -j200\ \Omega$$

$$Z_3 = j\left(1000 \cdot 50x10^{-3}\right) = +j50\ \Omega$$

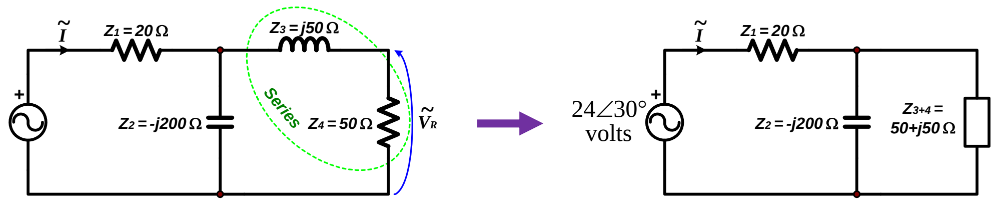
$$Z_4 = 50\ \Omega$$



Phasor Analysis Example #2

Step 2 – Reduce the network by combining Z_3 and Z_4 in series.

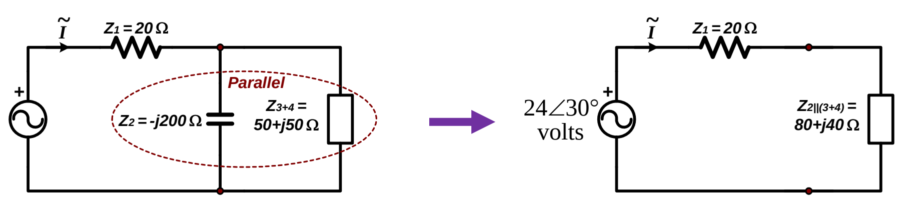
$$Z_{3+4} = Z_3 + Z_4 = 50 + j50\,\Omega$$



Phasor Analysis Example #2

Step 3 – Further reduce the network by combining Z_2 and Z_{3+4} in parallel.

$$Z_{2\|(3+4)} = \left(\frac{1}{Z_2} + \frac{1}{Z_{3+4}}\right)^{-1} = \left(\frac{1}{-j200} + \frac{1}{50 + j50}\right)^{-1} = 80 + j40\,\Omega$$



Phasor Analysis Example #2

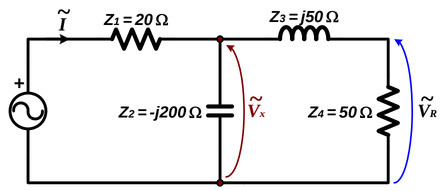
Step 4 – Solve for the voltage across $Z_{2||(3+4)}$.

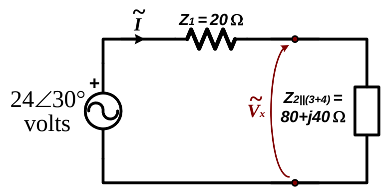
$$\tilde{V}_x = 24\angle 30° \cdot \frac{Z_{2||(3+4)}}{Z_1 + Z_{2||(3+4)}}$$

$$= 24\angle 30° \cdot \frac{80 + j40}{20 + 80 + j40}$$

$$= 19.9\angle 34.8° \text{ volts}$$

Note that the nodes across which $Z_{2||(3+4)}$ are connected also exist in the original circuit.





Phasor Analysis Example #2

Step 5 – Utilize V_x to determine the resistor voltage V_R.

$$\tilde{V}_R = \tilde{V}_x \cdot \frac{Z_4}{Z_3 + Z_4}$$

$$= 19.9\angle 34.8° \cdot \frac{50}{50 + j50}$$

$$= \boxed{14.1\angle -10.2° \text{ volts}}$$