ECET 2111

Circuits II

Introduction to Filters

Decibels & Logarithms

Decibel – a unit defined by a logarithmic expression that is commonly used to define the levels of a variety of parameters including voltage gain, field strength, energy, and sound pressure.

Logarithm – a quantity representing the power to which a fixed number (the base) must be raised to produce a given number.

Given: $A = B^x$

Then: $x = \log_B A$

Logarithms

Commonly used logarithms include:

$$x = \log_{10} A \qquad A = 10^x$$

$$x = \log_e A \qquad A = e^x$$

Notes: $\log_e A = 2.303 \cdot \log_{10} A$

$$\log_e A \equiv \ln A$$

Logarithms

For example:

if	then
$10^{-3} = 0.001$	$\log_{10} 0.001 = -3$
$10^{-2} = 0.01$	$\log_{10} 0.01 = -2$
$10^{-1} = 0.1$	$\log_{10} 0.1 = -1$
$10^{0} = 1$	$\log_{10} 1 = 0$
$10^{1} = 10$	$\log_{10} 10 = 1$
$10^{2} = 100$	$\log_{10} 100 = 2$
$10^{3} = 1000$	$\log_{10} 1000 = 3$

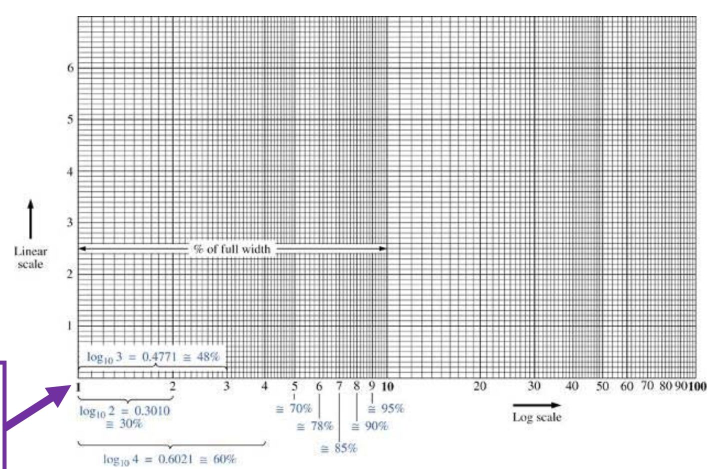
Logarithms

Logarithms are commonly utilized in order to:

- Plot the response of a system across a wide range of values that may be impractical if using a linear scale
- Compare levels of voltage or power without having to deal with very large or very small numbers that might otherwise obscure important details contained within the data
- Predict the operation of a system that has a non-linear response to stimuli in a logarithmic manner
- Determine the response of a cascaded or compound system provided that the gain of each stage is known in a logarithmic manner.

Semi-Log Plot

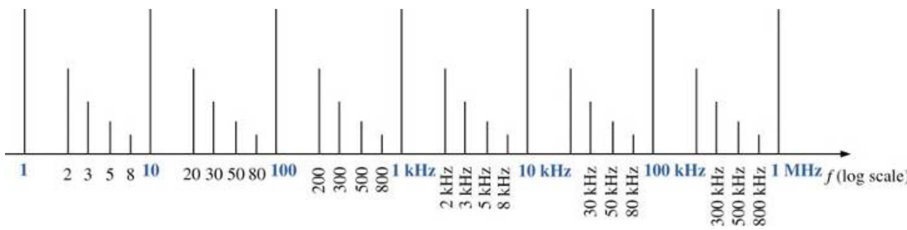
The following plot utilizes a logarithmic scale for its x-axis:



Note that when using a log-scale, the axis cannot begin at zero.

Logarithmic Scales

A semi-log plot is often utilized to display the frequency response of an electrical system.

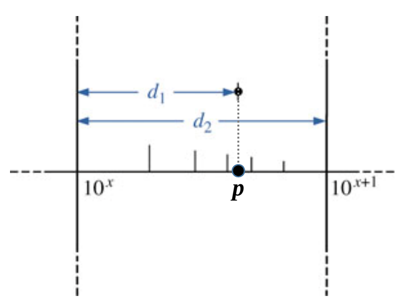


Logarithmic Scales

To find the specific value of a point plotted on a log-scale:

- Measure the distance, $\boldsymbol{d_1}$, between the plotted point and the closest major division to the left of the plotted point.
- Measure the distance, $\boldsymbol{d_2}$, between the closest major division to the left of the plotted point and the next major division.

$$p = 10^x \cdot 10^{\frac{d_1}{d_2}}$$



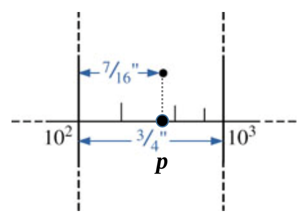
Logarithmic Scales

To find the specific value of a point plotted on a log-scale:

- Measure the distance, ***d*₁**, between the plotted point and the closest major division to the left of the plotted point.
- Measure the distance, ***d*₂**, between the closest major division to the left of the plotted point and the next major division.

$$p = 10^2 \cdot 10^{\frac{7/16}{3/4}}$$

$$= 383$$



Properties of Logarithms

- The Log of one (1) is always equal to zero (0).

$$\log_{10} 1 = 0 \qquad \log_e 1 = 0 \qquad \log_n 1 = 0$$

- If (A>1) then the Log of A is positive.

$$\log_{10} 2000 = 3.3 \qquad \log_e 5 = 1.61$$

- If (A<1) then the Log of A is negative.

$$\log_{10} 0.5 = -0.3 \qquad \log_e 0.1 = -2.3$$

- Additional properties include:

$$\log_n a \cdot b = \log_n a + \log_n b \qquad \log_n \frac{a}{b} = \log_n a - \log_n b \qquad \log_n a^b = b \cdot \log_n a$$

Bels & Decibels

Power Gain

Bel (***B***) – a base unit defined as a logarithmic ratio of powers:

$$B = \log_{10} \frac{P_2}{P_1}$$

Decibel (***dB***) – a logarithmic ratio of powers that is commonly utilized in order to define the **gain** (increase) in power P_2 compared to power P_1.

$$dB = 10 \cdot B = 10 \cdot \log_{10} \frac{P_2}{P_1}$$

Properties of Decibels

- If $P_2 = P_1$, then the decibel **gain is zero**.

$$10 \cdot \log_{10} \frac{5mW}{5mW} = 10 \cdot \log_{10} 1 = 10 \cdot 0 = 0 \text{ dB}$$

- If $P_2 > P_1$, then the decibel **gain is positive**.

$$10 \cdot \log_{10} \frac{20mW}{5mW} = 10 \cdot \log_{10} 4 = 10 \cdot 0.6 = +6 \text{ dB}$$

- If $P_2 < P_1$, then the decibel **gain is negative**.

$$10 \cdot \log_{10} \frac{1mW}{100mW} = 10 \cdot \log_{10} 0.01 = 10 \cdot (-2) = -20 \text{ dB}$$

Properties of Decibels

- If $P_2 = 2^n \cdot P_1$, then the decibel gain is $n \cdot$ (+3dB).

$$P_1 = 1\,mW \qquad P_1 = 32\,mW = 2^5 \cdot 1\,mW \qquad 10 \cdot \log_{10} \frac{32mW}{1mW} = +15\text{ dB} = 5 \cdot (+3\text{ dB})$$

- If $P_2 = \frac{1}{2}^n \cdot P_1$, then the decibel gain is $n \cdot$ (–3dB).

$$P_1 = 200\,W \qquad P_1 = 25\,W = \left(\tfrac{1}{2}\right)^3 \cdot 200\,W \qquad 10 \cdot \log_{10} \frac{25W}{200W} = -9\text{ dB} = 3 \cdot (-3\text{ dB})$$

- If $P_2 = 10^n \cdot P_1$, then the decibel gain is $n \cdot$ (+10dB).
- If $P_2 = 10^{-n} \cdot P_1$, then the decibel gain is $n \cdot$ (–10dB)

dBm

dBm – a specific value of power, relating to a power P_2 (mW), but expressed in terms of the decibel gain of P_2 compared to a reference power of 1mW.

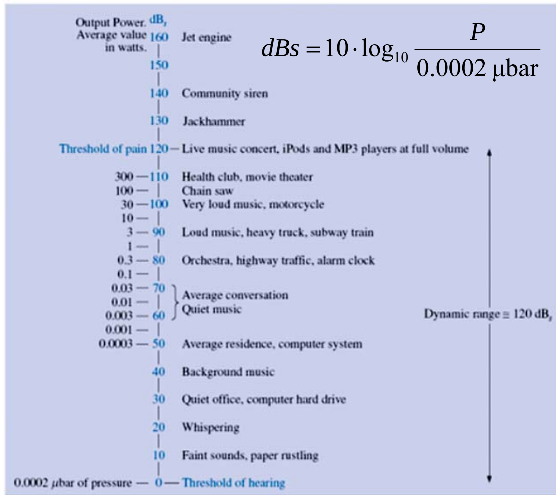
$$dBm = 10 \cdot \log_{10} \frac{P_2}{1\text{ mW}}$$

For example – convert a power of **+6dBm** to a **mW** value:

$$+6\,dBm = 10 \cdot \log_{10} \frac{P_2}{1\text{ mW}}$$

$$P_2 = 1\text{ mW} \cdot 10^{\frac{+6}{10}} = 1\text{ mW} \cdot 4 = \boxed{4\text{ mW}}$$

Decibel Example



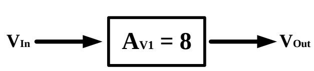
Typical Sound Levels and their Decibel Values

Voltage Gain

Voltage Gain (***A_V***) – a ratio of voltages that is commonly utilized in order to define the **gain** (increase) in voltage ***V_{Out}*** compared to voltage ***V_{In}***.

$$A_V = \frac{V_{Out}}{V_{In}}$$

For example – is an amplifier has a voltage gain ***A_V = 8***, then:



$$V_{Out} = A_V \cdot V_{In} = 8 \cdot V_{In}$$

dBv

dBv – a logarithmic ratio of voltages, expressed in terms of decibels, that is commonly utilized in order to define the **gain** in the power supplied to a resistive load ***R*** by voltage V_2 compared to the power supplied to the same resistive load ***R*** by voltage V_1.

$$dB = 10 \cdot \log_{10} \frac{P_2}{P_1} = 10 \cdot \log_{10} \frac{\frac{V_2^2}{R}}{\frac{V_1^2}{R}} = 10 \cdot \log_{10} \frac{V_2^2}{V_1^2} = 10 \cdot \log_{10} \left(\frac{V_2}{V_1} \right)^2 = 20 \cdot \log_{10} \frac{V_2}{V_1}$$

$$dB_V = 20 \cdot \log_{10} \frac{V_2}{V_1}$$

dBv

For example:

V_o/V_i	$dB = 20 \log_{10} (V_o/V_i)$
1	0 dB
2	6 dB
10	20 dB
20	26 dB
100	40 dB
1,000	60 dB
100,000	100 dB

$$dB_V = 20 \cdot \log_{10} \frac{V_2}{V_1}$$

Filters

A **filter** is a combination of elements that is designed to select or reject a band of frequencies.

Passive Filters are filters composed of series and/or parallel combinations of passive elements (R, L, and C).

Active Filters are filters that employ active electronic devices, such as transistors or operational amplifiers, in combination with passive elements.

Filters

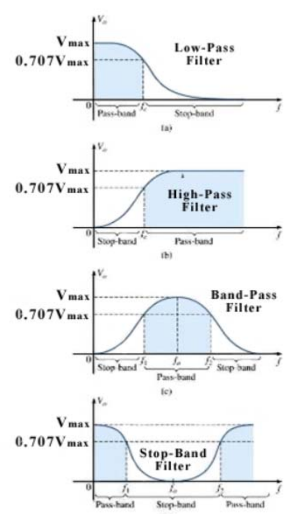
There are four primary categories of filters:

Low-Pass Filters

High-Pass Filters

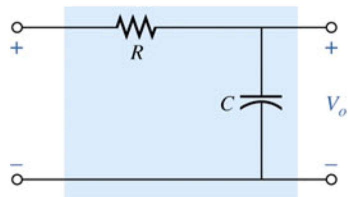
Band-Pass Filters

Stop-Band Filters



R-C Low-Pass Filter

The following combination of elements can be utilized to create a **R-C Low-Pass Filter**:

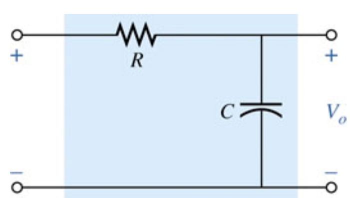


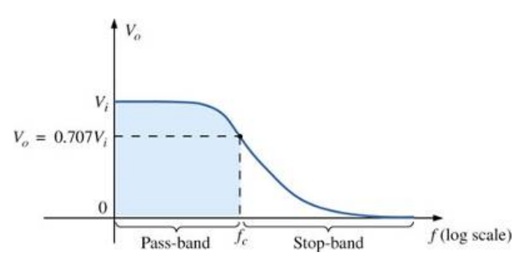
$$V_{out} = V_{in} \cdot \frac{-jX_C}{R - jX_C}$$

$$A_V = \frac{V_{out}}{V_{in}} = \frac{-jX_C}{R - jX_C}$$

R-C Low-Pass Filter

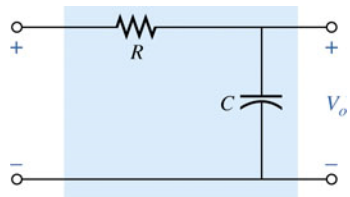
The R-C Low-Pass Filter has the **frequency response** shown to the right:

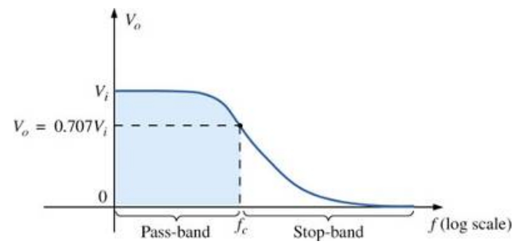




R-C Low-Pass Filter

The **cutoff frequency** for the filter is defined to be the frequency at which the output voltage is 0.707x its peak value.

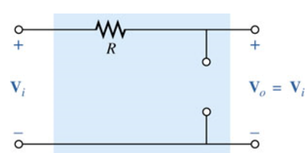




$$f_c = \frac{1}{2 \cdot \pi \cdot R \cdot C}$$

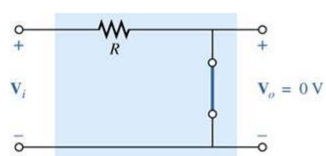
R-C Low-Pass Filter

At low frequencies, the capacitor acts like an open-circuit:



$$V_{out} = V_{in}$$

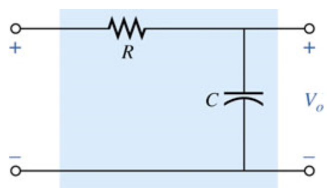
while at high frequencies, the capacitor acts like a short-circuit:

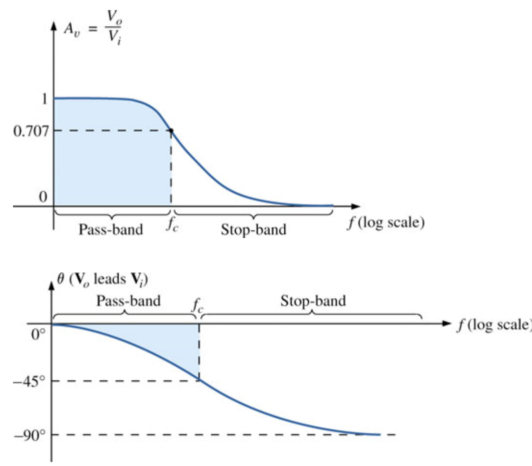


$$V_{out} = 0$$

R-C Low-Pass Filter

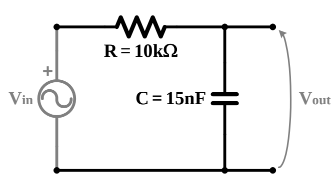
The following plots show the **voltage gain** and **phase response** of the low-pass filter:





R-C Low-Pass Filter Example

Determine the voltage gain and phase response of the following **low-pass filter**.



The OC output voltage and voltage gain are defined by:

$$V_{out} = V_{in} \cdot \frac{-jX_C}{R - jX_C}$$

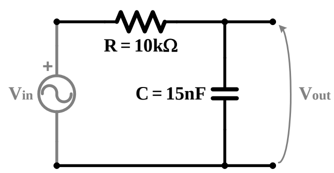
$$A_V = \frac{V_{out}}{V_{in}} = \frac{-jX_C}{R - jX_C}$$

and the cutoff frequency is:

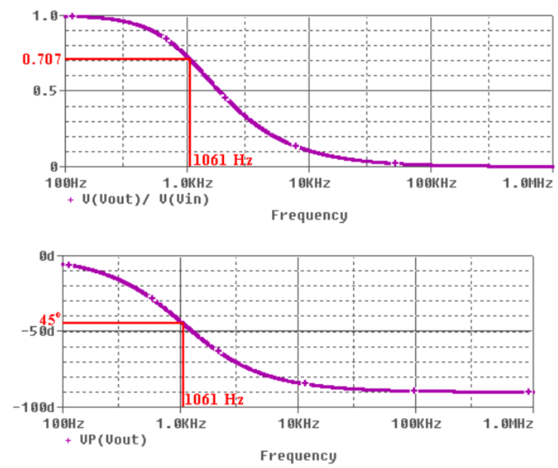
$$f_c = \frac{1}{2 \cdot \pi \cdot R \cdot C} = \frac{1}{2 \cdot \pi \cdot 10000 \cdot 15 \cdot 10^{-9}} = 1061\,\text{Hz}$$

R-C Low-Pass Filter Example

The **voltage gain** and **phase response** plots for the circuit are:

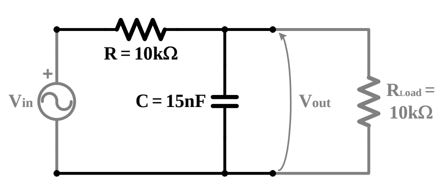


$$f_c = \frac{1}{2\cdot\pi\cdot R\cdot C} = 1061\,\text{Hz}$$



R-C Low-Pass Filter Example

Determine the response of the low-pass filter if a **10kΩ load** is added to the circuit:



The OC output voltage and voltage gain are defined by:

$$V_{out} = V_{in}\cdot\frac{Z_p}{R+Z_p}$$

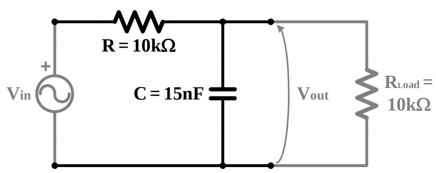
$$A_V = \frac{V_{out}}{V_{in}} = \frac{Z_p}{R+Z_p}$$

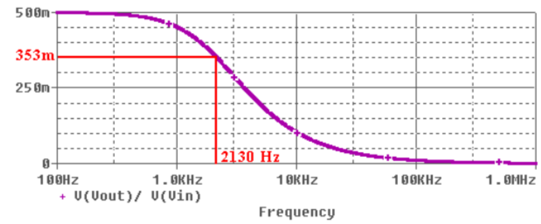
where:

$$Z_p = -jX_C \parallel R_{Load}$$

R-C Low-Pass Filter Example

The **voltage gain** for the circuit with the **10kΩ load** is:





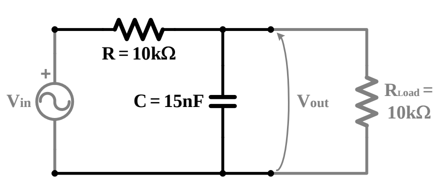
$f_{c(OC)} \approx 1061$ Hz

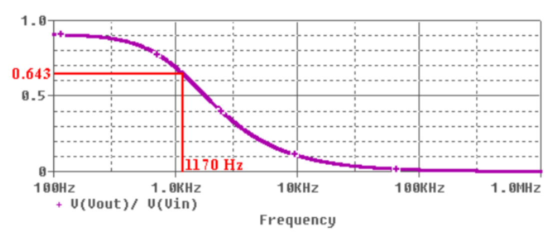
$f_{c(10\text{k Load})} \approx 2130$ Hz

Notice that the cutoff frequency has increased from 1061Hz when open-circuited to 2130Hz with the 10kHz load.

R-C Low-Pass Filter Example

If the load resistance is increased to **100kΩ**, the **voltage gain** for the circuit is:





$f_{c(OC)} \approx 1061$ Hz

$f_{c(10\text{k Load})} \approx 2130$ Hz

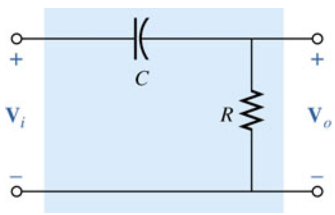
$f_{c(100\text{k Load})} \approx 1170$ Hz

Notice that the cutoff frequency has decreased to 1170Hz.

If $R_{load} >> R$, the response of the filter will resemble the filter's OC response.

R-C High-Pass Filter

The following combination of elements can be utilized to create a **R-C High-Pass Filter**:

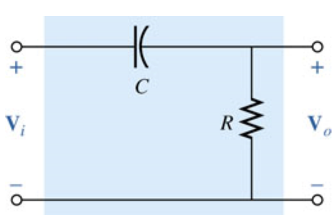


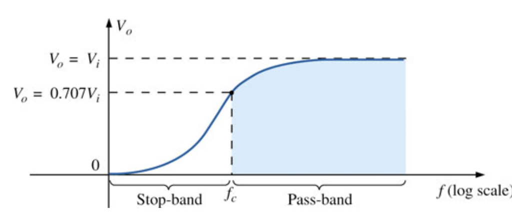
$$V_{out} = V_{in} \cdot \frac{R}{R - jX_C}$$

$$A_V = \frac{V_{out}}{V_{in}} = \frac{R}{R - jX_C}$$

R-C High-Pass Filter

The R-C High-Pass Filter has the **frequency response** shown to the right, the **cutoff frequency** for which is also defined to be the frequency at which the output voltage is 0.707x its peak value:

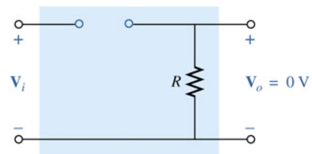




$$f_c = \frac{1}{2 \cdot \pi \cdot R \cdot C}$$

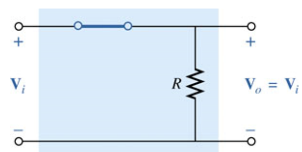
R-C High-Pass Filter

At low frequencies, the capacitor acts like an open-circuit:



$$V_{out} = 0$$

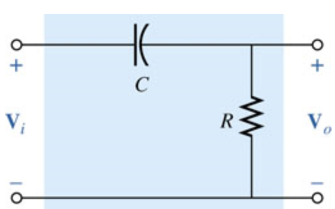
while at high frequencies, the capacitor acts like a short-circuit:

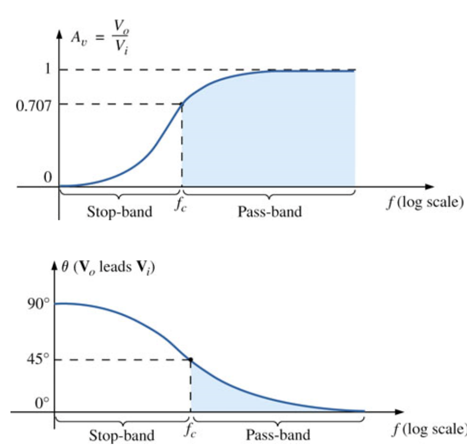


$$V_{out} = V_{in}$$

R-C High-Pass Filter

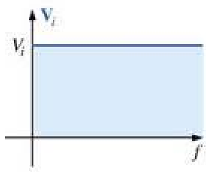
The following plots show the **voltage gain** and **phase response** of the high-pass filter:

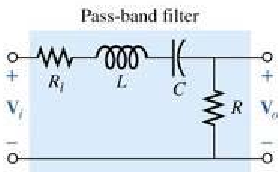


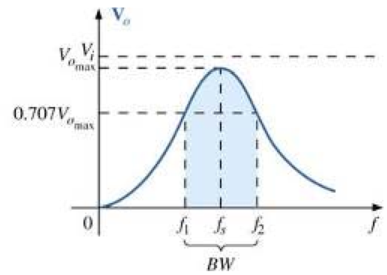


Band-Pass Filter

The following combination of elements can be utilized to create a series-resonant **Band-Pass Filter**:





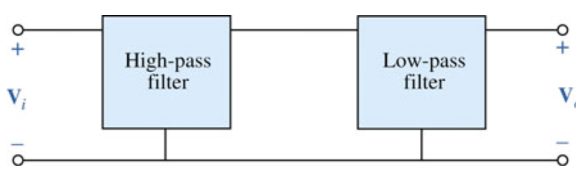


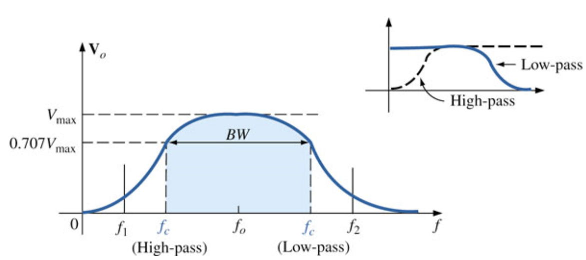
$$f_s = \frac{1}{2 \cdot \pi \cdot \sqrt{L \cdot C}}$$

$$V_{out} = V_{in} \cdot \frac{R}{R + R_L + jX_L - jX_C} \qquad A_V = \frac{V_{out}}{V_{in}} = \frac{R}{R + R_L + jX_L - jX_C}$$

Band-Pass Filter

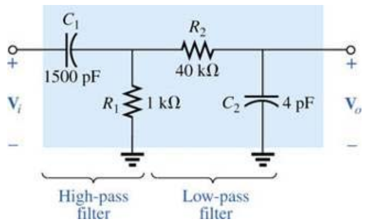
A **Band-Pass Filter** could also be created by **cascading** a High-Pass Filter and a Low-Pass Filter:

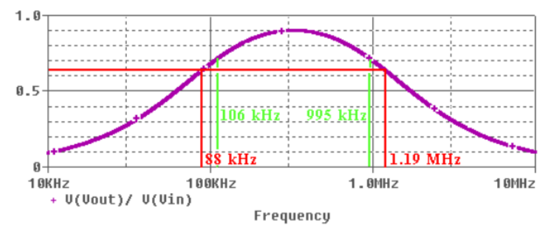




Band-Pass Filter Example

Note that, when cascading a High-Pass Filter and a Low-Pass Filter to create a **Band-Pass Filter**, the actual cutoff frequencies will vary from the individual cutoff frequencies due to the interaction between the cascaded filters.





$$f_{c(HP)} = \frac{1}{2\cdot\pi\cdot R_1\cdot C_1} = 106\ kHz$$

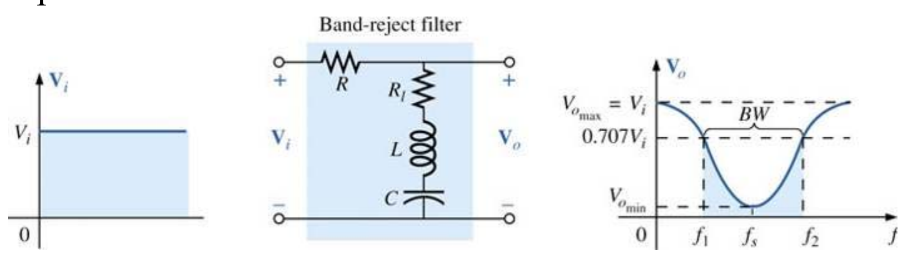
$$f_{c(LP)} = \frac{1}{2\cdot\pi\cdot R_2\cdot C_2} = 995\ kHz$$

$$f_{lc(actual)} \approx 88\ kHz$$

$$f_{uc(actual)} \approx 1190\ kHz$$

Band-Stop Filter

A series-resonant **Band-Stop Filter** can be created by reversing the positions of the elements used in the Band-Pass Filter:



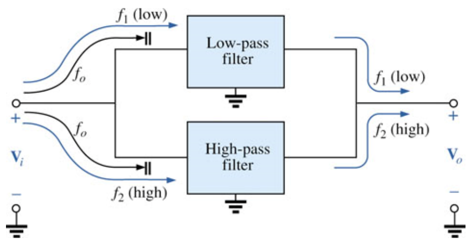
$$f_s = \frac{1}{2\cdot\pi\cdot\sqrt{L\cdot C}}$$

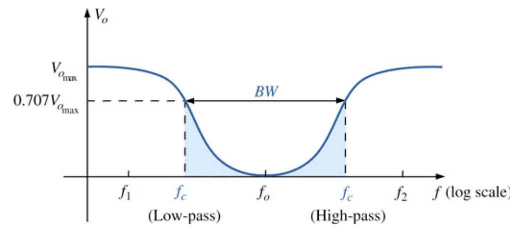
$$V_{out} = V_{in}\cdot\frac{R_L + jX_L - jX_C}{R + R_L + jX_L - jX_C}$$

$$A_V = \frac{V_{out}}{V_{in}} = \frac{R_L + jX_L - jX_C}{R + R_L + jX_L - jX_C}$$

Band-Stop Filter

A **Band-Stop Filter** could also be created by **paralleling** a High-Pass Filter and a Low-Pass Filter:





Bode Plots

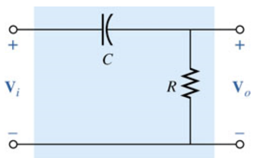
<u>Bode Plots</u> are the curves obtained for the magnitude and phase response (versus frequency) of a system.

Idealized Bode Plots utilize straight-line segments to efficiently estimate the frequency response of a system.

There is a quick technique for sketching the frequency response of a system on a decibel scale that provides a good method for comparing the expected decibel levels at different frequencies.

R-C High-Pass Filter

Lets look back at the **R-C High-Pass Filter**.



$$f_c = \frac{1}{2 \cdot \pi \cdot R \cdot C}$$

The formula for the voltage gain can be rewritten as:

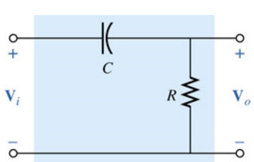
$$A_V = \frac{V_{out}}{V_{in}} = \frac{R}{R - jX_C}$$

$$= \frac{1}{1 - j\dfrac{X_C}{R}}$$

$$= \frac{1}{1 - j\dfrac{1}{2\pi \cdot f \cdot R \cdot C}}$$

R-C High-Pass Filter

Lets look back at the **R-C High-Pass Filter**.



$$f_c = \frac{1}{2\pi \cdot R \cdot C}$$

Additionally, The formula for the voltage gain can be rewritten as:

$$A_V = \frac{V_{out}}{V_{in}} = \frac{1}{1 - j\dfrac{1}{2\pi \cdot f \cdot R \cdot C}}$$

$$= \frac{1}{1 - j\left(\dfrac{1}{2\pi \cdot R \cdot C}\right) \cdot \dfrac{1}{f}}$$

$$= \frac{1}{1 - j\dfrac{f_c}{f}}$$

R-C High-Pass Filter

Given the voltage gain for a **R-C High-Pass Filter**:

$$A_V = \frac{V_{out}}{V_{in}} = \frac{1}{1 - j\frac{f_c}{f}}$$

the **magnitude** of the voltage gain can be expressed as:

$$|A_V| = \frac{1}{\sqrt{1 + \left(\frac{f_c}{f}\right)^2}} = \frac{1}{\left(1 + \left(\frac{f_c}{f}\right)^2\right)^{\frac{1}{2}}}$$

R-C High-Pass Filter

If the **magnitude** of the voltage gain is expressed in **decibels**:

$$|A_V|_{dB} = 20\log_{10}|A_V|$$

the result will be:

$$|A_V|_{dB} = 20\log_{10}\frac{1}{\left(1 + \left(\frac{f_c}{f}\right)^2\right)^{\frac{1}{2}}}$$

$$= 20\log_{10}1 - 20\log_{10}\left(1 + \left(\frac{f_c}{f}\right)^2\right)^{\frac{1}{2}}$$

$$= 0 - 20\log_{10}\left(1 + \left(\frac{f_c}{f}\right)^2\right)^{\frac{1}{2}}$$

R-C High-Pass Filter

Additionally, the **decibel voltage gain** can be rewritten as:

$$\left|A_V\right|_{dB} = -20\log_{10}\left(1+\left(\frac{f_c}{f}\right)^2\right)^{\frac{1}{2}}$$

$$= -\tfrac{1}{2}\cdot 20\log_{10}\left(1+\left(\frac{f_c}{f}\right)^2\right)$$

$$= -10\log_{10}\left(1+\left(\frac{f_c}{f}\right)^2\right)$$

R-C High-Pass Filter

Given the **decibel voltage gain** function:

$$\left|A_V\right|_{dB} = -10\log_{10}\left(1+\left(\frac{f_c}{f}\right)^2\right)$$

when $\boldsymbol{f << f_c}$,

$$1+\left(\frac{f_c}{f}\right)^2 \cong \left(\frac{f_c}{f}\right)^2$$

and:

$$\left|A_V\right|_{dB(f<<f_c)} = -10\log_{10}\left(\frac{f_c}{f}\right)^2 = -20\log_{10}\left(\frac{f_c}{f}\right) = +20\log_{10}\left(\frac{f}{f_c}\right)$$

Bode Plots

Thus, given the **decibel voltage gain** function ($\boldsymbol{f \ll f_c}$):

$$\left|A_V\right|_{dB(f \ll f_c)} = +20\log_{10}\left(\frac{f}{f_c}\right)$$

if	**then**
$f = f_c$	**20log$_{10}$(1) = 0dB**
$f = 0.5f_c$	**20log$_{10}$(0.5) = -6dB**
$f = 0.25f_c$	**20log$_{10}$(0.25) = -12dB**
$f = 0.1f_c$	**20log$_{10}$(0.1) = -20d**

Bode Plots

Note that, given the **decibel voltage gain** function ($\boldsymbol{f \ll f_c}$):

$$\left|A_V\right|_{dB(f \ll f_c)} = +20\log_{10}\left(\frac{f}{f_c}\right)$$

For every decrease in the frequency by a factor of 0.5 (one octave), there will be a 6dB decrease in the gain, and

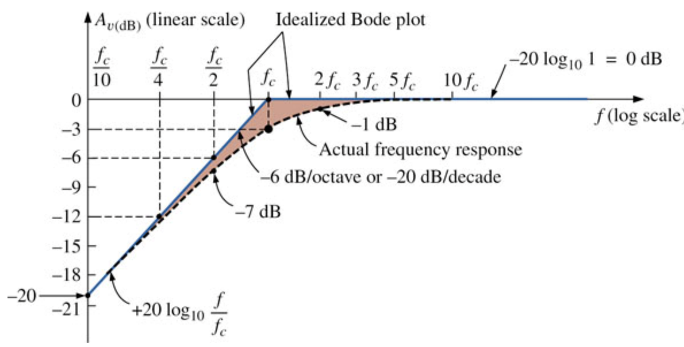
For every decrease in the frequency by a factor of 0.1 (one decade), there will be a 20dB decrease in the gain.

Thus, an **Idealized Bode Plot** can be drawn for the gain function because the dB change per octave or decade is constant.

Bode Plots

The Bode Plot for the **decibel voltage gain** function is:

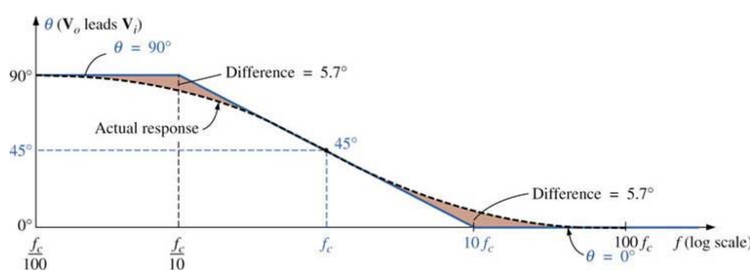
$$|A_V|_{dB(f<<f_c)} = +20\log_{10}\left(\frac{f}{f_c}\right)$$



Bode Plots

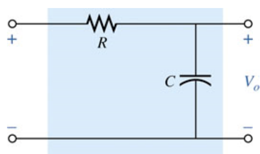
Additionally, the phase response may be drawn as:

$$\theta = \tan^{-1}\left(\frac{f_c}{f}\right)$$



R-C Low-Pass Filter

Given an **R-C Low-Pass Filter**, the **decibel voltage gain** ($f >> f_c$) can be written as:

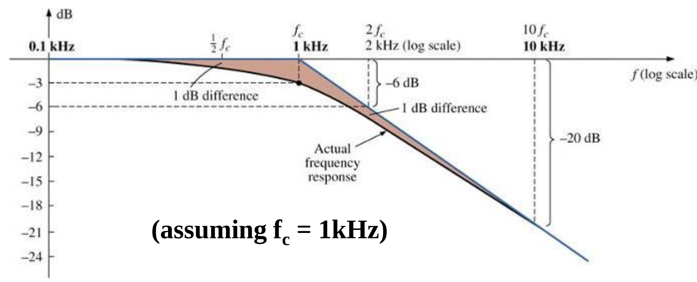


$$f_c = \frac{1}{2 \cdot \pi \cdot R \cdot C}$$

$$A_V = \frac{V_{out}}{V_{in}} = \frac{1}{1 + j\,\frac{f_c}{f}}$$

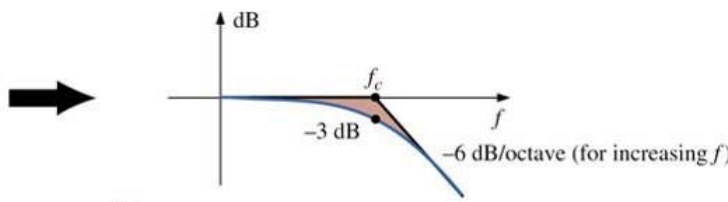
$$\left|A_V\right|_{dB(f >> f_c)} = -20\log_{10}\left(\frac{f}{f_c}\right)$$

resulting in the following **Bode Plot**:



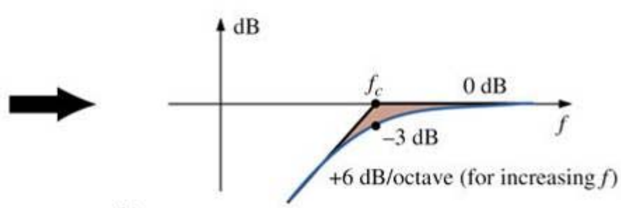
Sketching Bode Plots

Low-pass: $\dfrac{1}{1 + j\,\dfrac{f}{f_c}}$



(a)

High-pass: $\dfrac{1}{1 + j\,\dfrac{f_c}{f}}$



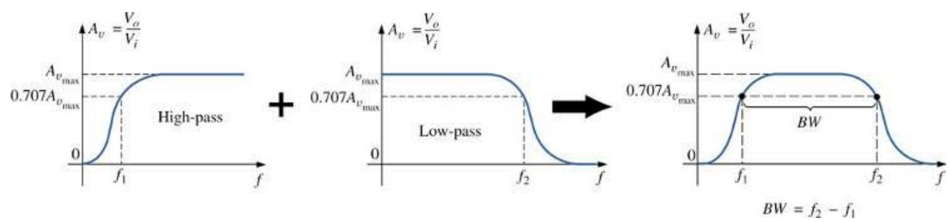
(b)

Sketching Bode Plots Example

Given:

$$A_V = \frac{1}{\left(1 - j\dfrac{50\text{ Hz}}{f}\right)\cdot\left(1 - j\dfrac{200\text{ Hz}}{f}\right)\cdot\left(1 + j\dfrac{f}{10\text{ kHz}}\right)\cdot\left(1 + j\dfrac{f}{20\text{ kHz}}\right)}$$

HPF + HPF + LPF + LPF

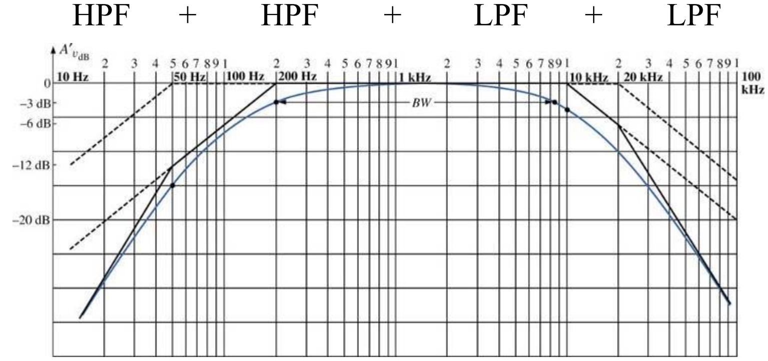


Sketching Bode Plots Example

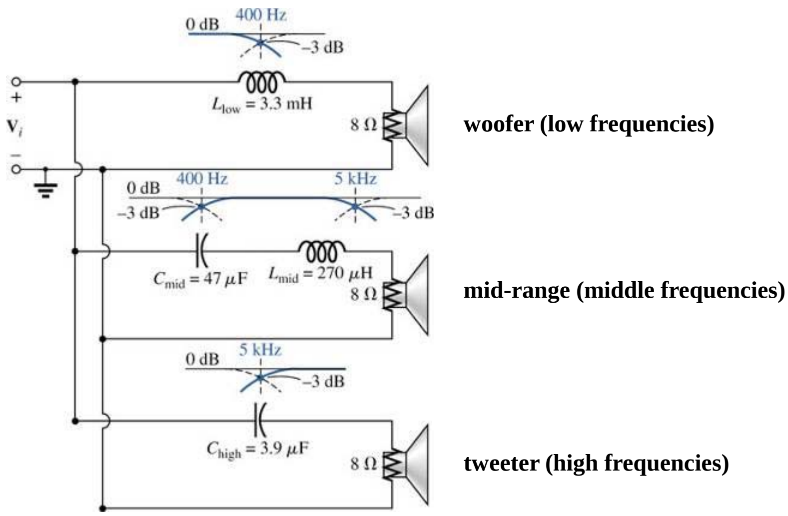
Given:

$$A_V = \frac{1}{\left(1 - j\dfrac{50\text{ Hz}}{f}\right)\cdot\left(1 - j\dfrac{200\text{ Hz}}{f}\right)\cdot\left(1 + j\dfrac{f}{10\text{ kHz}}\right)\cdot\left(1 + j\dfrac{f}{20\text{ kHz}}\right)}$$

HPF + HPF + LPF + LPF



3-Way Crossover w/ 6dB per Octave



Idealized Bode Plots for Various Functions