ECET 2111

Circuits II

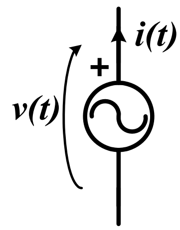
Complex Power

in

Steady-State AC Circuits

Steady-State AC Voltage Sources

The voltage potential of an AC source may be defined as:

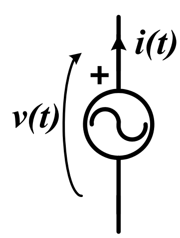


$$v(t) = V_{peak} \cdot \sin(\omega \cdot t + \phi)$$

where: V_{peak} **is the peak value of the voltage,**
ω **is the angular frequency ($2\pi f$) of the waveform, and**
ϕ **is the phase angle of the voltage waveform.**

Steady-State AC Current Sources

Similarly, the current produced by the AC source may be defined as:



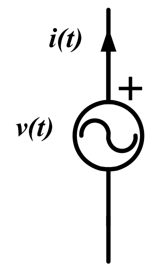
$$i(t) = I_{peak} \cdot \sin(\omega \cdot t + \delta)$$

where: I_{peak} **is the peak value of the current,**
ω **is the angular frequency ($2\pi f$) of the waveform, and**
δ **is the phase angle of the current waveform.**

Power in AC Circuits

In electric circuits, **power** can be defined as the **rate** at which **electric energy** is either produced or consumed by an element within the circuit.

Although it is actually the **electric energy** that is either being produced or consumed by the circuit elements, power is also casually referred to as being produced or consumed within an electric circuit.

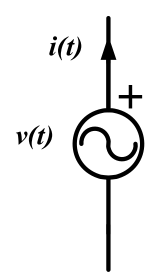


Power in AC Circuits

Power may be calculated in terms of the voltage and current waveforms associated with a specific circuit element by:

$$p(t) = v(t) \cdot i(t) \quad \textit{(Watts)}$$

where: $p(t)$ provides the **instantaneous rate** that an element either produces or consumes electric energy at any time *t*.

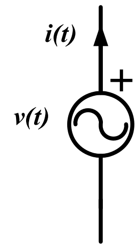


Power in AC Circuits

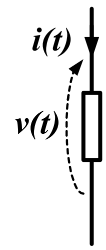
Note that the expression:

$$p(t) = v(t) \cdot i(t) \quad \textit{(Watts)}$$

defines the power "**produced**" by an element when the current is defined in the <u>same direction</u> as the voltage-rise across the element.



But, if the current is defined in the <u>opposite direction</u> as the voltage-rise across an element, then *p(t)* defines the power "**consumed**" by that element.

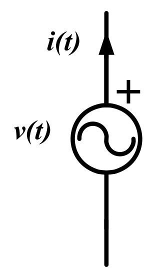


Power from an AC Source

In the case of an AC source where:

$$v(t) = V_{peak} \cdot \sin(\omega \cdot t + \phi)$$

$$i(t) = I_{peak} \cdot \sin(\omega \cdot t + \delta)$$



the general expression for **power** produced by the source is:

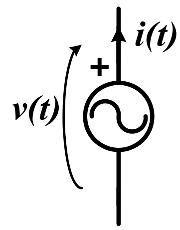
$$p(t) = v(t) \cdot i(t)$$
$$= V_{peak} \cdot I_{peak} \cdot \sin(\omega \cdot t + \phi) \cdot \sin(\omega \cdot t + \delta)$$

Power from an AC Source

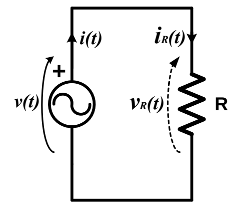
The **power** expression:

$$p(t) = V_{peak} \cdot I_{peak} \cdot \sin(\omega \cdot t + \phi) \cdot \sin(\omega \cdot t + \delta)$$

is actually quite complex.



To better understand the true nature of the power expression, it may by useful to first consider the case where the voltage source is applied to a **purely resistive load**.



AC Sources and Resistive Loads

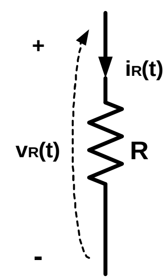
Given a resistor, whose voltage is:

$$v_R(t) = V_{peak} \cdot \sin(\omega \cdot t + \phi)$$

the resultant resistor current will be:

$$i_R(t) = \frac{v_R(t)}{R} = \frac{V_{peak} \cdot \sin(\omega \cdot t + \phi)}{R}$$

$$= \frac{V_{peak}}{R} \cdot \sin(\omega \cdot t + \phi)$$

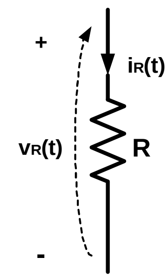


AC Sources and Resistive Loads

Thus, for a resistive load:

$$v_R(t) = \boxed{V_{peak}} \cdot \sin(\omega \cdot t + \phi)$$

$$i_R(t) = \boxed{\frac{V_{peak}}{R}} \cdot \sin(\omega \cdot t + \phi)$$



Note that, the voltage and current **magnitudes** follow the Ohm's Law relationship:

$$I_{peak} = \frac{V_{peak}}{R} \ ,$$

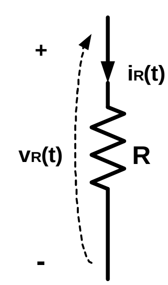
and that the sinusoidal expression remains unchanged.

AC Sources and Resistive Loads

Thus, for a resistive load:

$$v_R(t) = V_{peak} \cdot \sin(\omega \cdot t + \phi)$$

$$i_R(t) = \frac{V_{peak}}{R} \cdot \sin(\omega \cdot t + \phi)$$

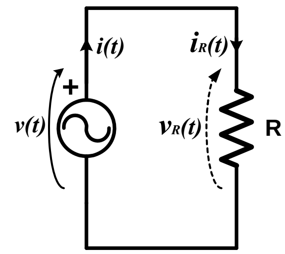


Based on this result, it can be seen that both the **frequency** and the **phase angle** of the resistor current are equal to those of the applied voltage…

For this reason, AC circuits containing resistive loads are often analyzed in terms of the magnitudes of the voltages and currents.

AC Power and Resistors

If an AC source is connected to a **resistive load**, such that:



$$v_R(t) = V_{peak} \cdot \sin(\omega \cdot t + \phi)$$

$$i_R(t) = I_{peak} \cdot \sin(\omega \cdot t + \phi)$$

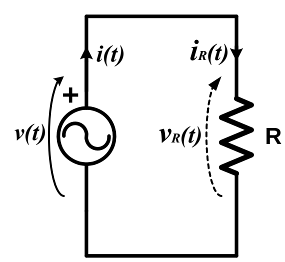
$$I_{peak} = \frac{V_{peak}}{R}$$

then the **power** consumed by the resistor will be:

$$p_R(t) = v_R(t) \cdot i_R(t)$$
$$= V_{peak} \cdot I_{peak} \cdot \sin^2(\omega \cdot t + \phi)$$

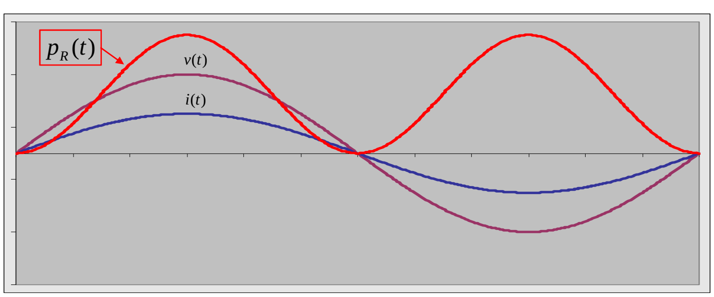
AC Power and Resistors

The figure below shows the **power waveform**:



$$p_R(t) = V_{peak} \cdot I_{peak} \cdot \sin^2(\omega \cdot t + \phi)$$

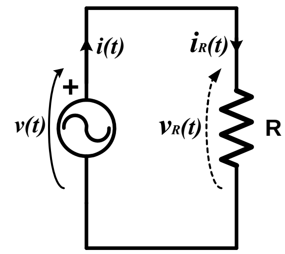
plotted along with the resistor's voltage and current waveforms:

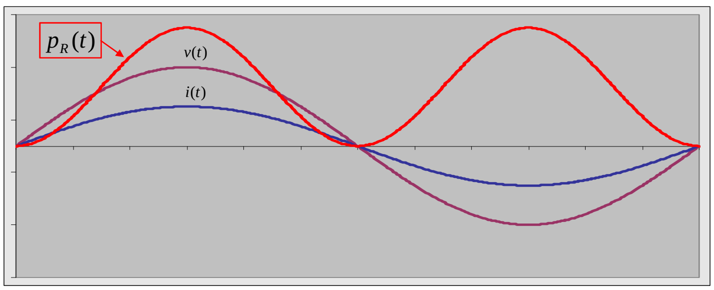


AC Power and Resistors

As shown, **power supplied to the resistor** is always non-negative, which is expected since a resistor can only consume electric power.

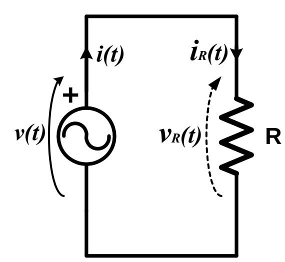
Note – Since *p(t)* is the power "consumed" by the resistor, a negative value would imply that power is actually being "produced" by the resistor, a result which can not occur.

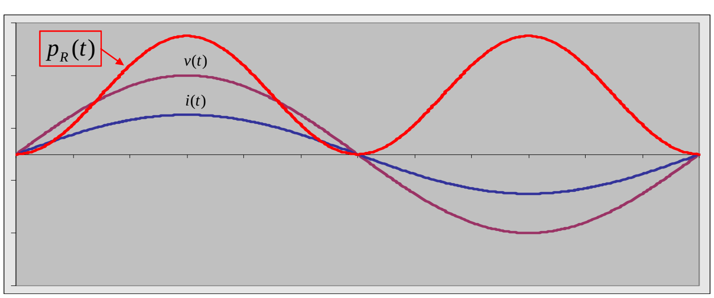




AC Power and Resistors

Additionally, it can be seen that the power waveform varies **periodically**, but with a **frequency** that is **2x larger** than that of the applied voltage or the resultant current.



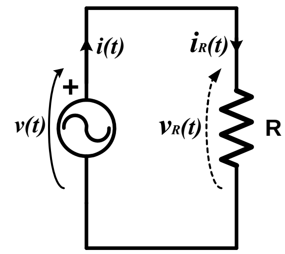


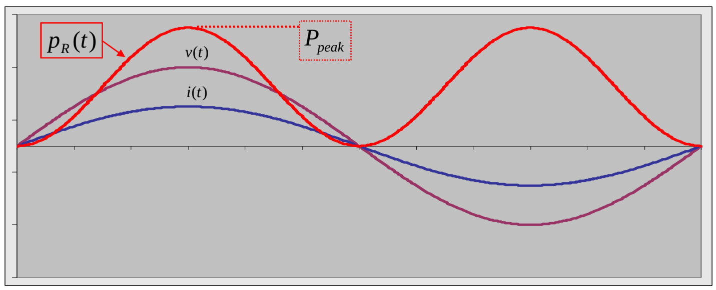
AC Power and Resistors

The peak magnitude of the **AC** power waveform is:

$$P_{peak} = V_{peak} \cdot I_{peak}$$

This should not be confused with the constant power provided to a resistor by a DC source.

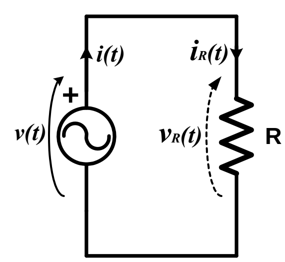




AC Power and Resistors

To better understand the resistor's AC power waveform, it is useful to rewrite the power expression:

(by utilizing the trig-identity $sin^2 x = ½·[1 – cos2x]$):



$$p_R(t) = V_{peak} \cdot I_{peak} \cdot \sin^2(\omega \cdot t + \phi)$$

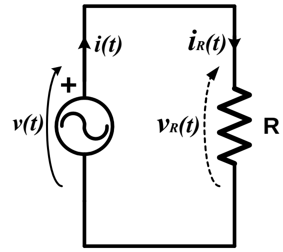
$$= \frac{V_{peak} \cdot I_{peak}}{2} \cdot [1 - \cos(2 \cdot \omega \cdot t)]$$

$$= \boxed{\frac{V_{peak} \cdot I_{peak}}{2} - \frac{V_{peak} \cdot I_{peak}}{2} \cdot \cos(2 \cdot \omega \cdot t)}$$

AC Power and Resistors

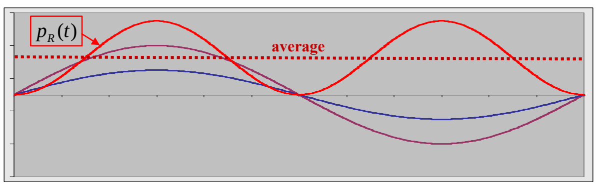
Looking at the resultant AC power waveform:

$$p_R(t) = \boxed{\frac{V_{peak} \cdot I_{peak}}{2}} - \frac{V_{peak} \cdot I_{peak}}{2} \cdot \cos(2 \cdot \omega \cdot t)$$



It can be seen that the waveform has two terms:

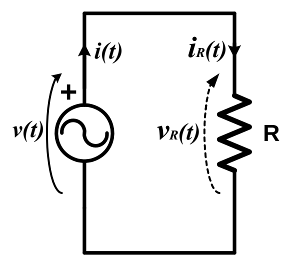
- The **first term** is a constant that relates to the **average** value of the power that is consumed by the resistor.



AC Power and Resistors

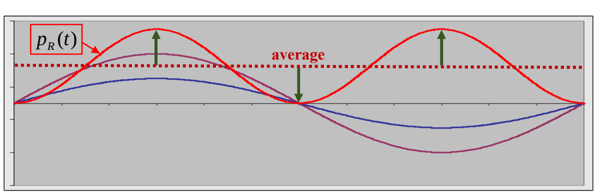
Looking at the resultant AC power waveform:

$$p_R(t) = \frac{V_{peak} \cdot I_{peak}}{2} - \boxed{\frac{V_{peak} \cdot I_{peak}}{2} \cdot \cos(2 \cdot \omega \cdot t)}$$



It can be seen that the waveform has two terms:

- The **second term** is a sinusoidal term that varies at 2x the source frequency and provides the fluctuation in the power waveform.



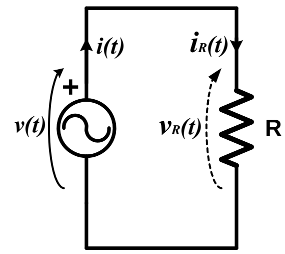
Real Power

In AC systems, it is typically the **average value** of the power that is desired.

This **average power** value is called **<u>Real Power</u>**.

The **real power** consumed by a resistive load is:

$$P_{R(AC)} = Avg[p_R(t)] = \frac{V_{peak} \cdot I_{peak}}{2} \quad \textit{(Watts)}$$



AC vs. DC Power to Resistors

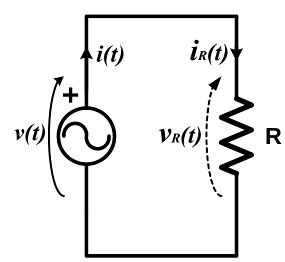
Note that the **real power** consumed by the resistor is <u>½ that of the peak power</u> value:

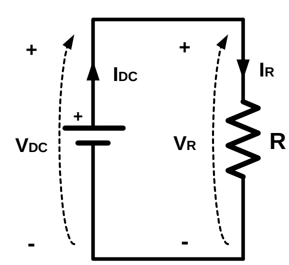
$$P_{R(AC)} = \frac{P_{peak}}{2} = \frac{V_{peak} \cdot I_{peak}}{2} \quad \textit{(Watts)}$$

This result is expected since the power waveform fluctuates evenly between zero and its peak value.

Yet, this result is potentially confusing if compared to <u>power supplied by a DC source</u> to a resistor:

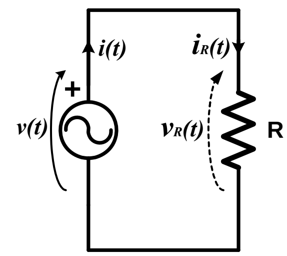
$$P_{R(DC)} = V_{DC} \cdot I_{DC} \quad \textit{(Watts)}$$





AC vs. DC Power to Resistors

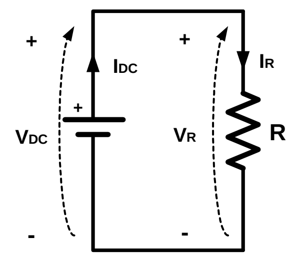
Thus, in terms of the **average power** supplied to a resistor, <u>an AC source is only ½ as **effective** as a DC source</u> whose magnitude is equal to the peak value of the AC source.



$$P_{R(AC)} = \frac{V_{peak} \cdot I_{peak}}{2} \quad (Watts)$$

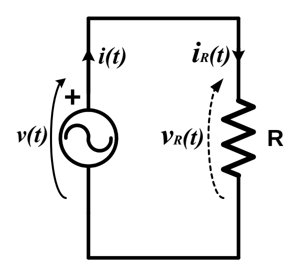
If: $V_{peak} = V_{DC}$ →

$$P_{R(DC)} = V_{DC} \cdot I_{DC} \quad (Watts)$$



Effective Voltage

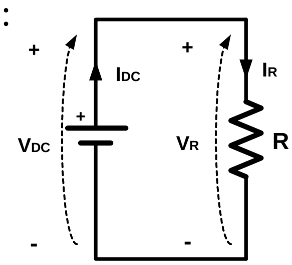
Since the **average AC power** is proportional to the square of the source's peak voltage:



$$P_{R(AC)} = \frac{V_{peak} \cdot I_{peak}}{2} = \frac{V_{peak} \cdot V_{peak}}{2 \cdot R} = \frac{V_{peak}^2}{2 \cdot R}$$

if the peak value of the AC voltage is increased such that it is $\sqrt{2}$ **times larger** than the DC voltage:

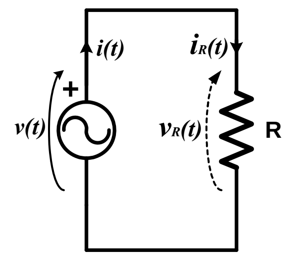
$$V_{peak} = \sqrt{2} \cdot V_{DC}$$



then the AC source will supply the **same average power** to the resistor as the DC source.

Effective / RMS Voltage Magnitudes

Based on this result, an **effective voltage** can be defined for a sinusoidally-varying AC source, such that:



$$V_{effective} = \frac{V_{peak}}{\sqrt{2}}$$

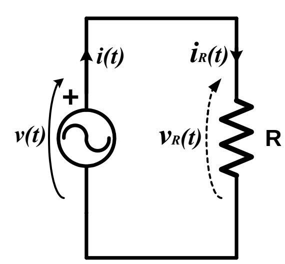
Note that the effective value of the AC source is equal to the **RMS** (***root-mean-squared***) value of the source voltage, as defined by the function:

$$V_{effective} = V_{RMS} = \sqrt{\frac{1}{T} \cdot \int_0^T v^2(t) \cdot dt} = \frac{V_{peak}}{\sqrt{2}}$$

Effective Voltage

For example:

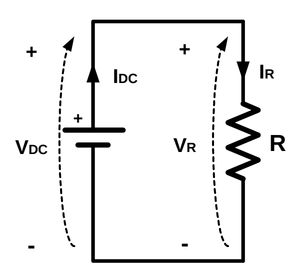
A 100V_{peak} AC source has an **effective voltage** of:



$$V_{effective} = V_{RMS} = \frac{V_{peak}}{\sqrt{2}} = \frac{100}{\sqrt{2}} = 70.7\ volts$$

since it delivers an average power of 50W to a 100Ω resistor:

$$P_{R(AC)} = \frac{V_{peak}^2}{2 \cdot R} = \frac{100^2}{2 \cdot 100} = 50\ Watts$$

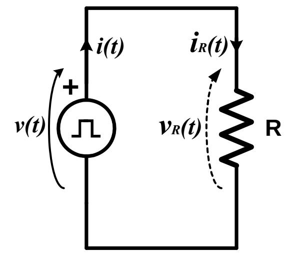


which is equal to that from a 70.7V DC source:

$$P_{R(DC)} = \frac{V_{DC}^2}{R} = \frac{70.7^2}{100} = 50\ Watts$$

Effective / RMS Voltage Magnitudes

It also turns out that the **effective voltage** for any non-sinusoidal, periodic AC voltage waveform, ***v(t)***, can be determined by calculating the waveform's RMS value:

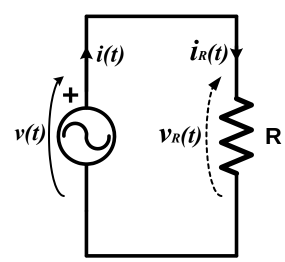


$$V_{RMS} = \sqrt{\frac{1}{T} \cdot \int_0^T v^2(t) \cdot dt}$$

Despite this fact, we will limit our discussion to sinusoidally-varying voltages.

RMS Magnitudes

The **voltage waveform** may be expressed in terms of its ***RMS voltage magnitude***:

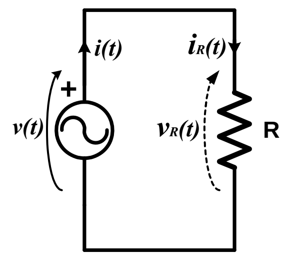


$$v(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

where: $V = \frac{V_{peak}}{\sqrt{2}}$ is the RMS magnitude of the AC voltage.

RMS Magnitudes

Similarly, the **current waveform** may also be expressed in terms of its ***RMS current magnitude***:

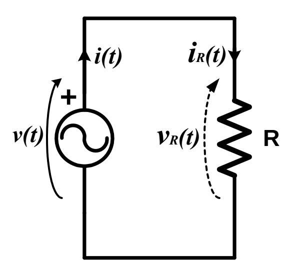


$$i(t) = \sqrt{2} \cdot I \cdot \sin(\omega \cdot t + \phi)$$

where: $I = \frac{I_{peak}}{\sqrt{2}}$ is the RMS magnitude of the AC current.

RMS Magnitudes & Resistor Power

When the voltages and currents are expressed in terms of their **RMS magnitudes**:



$$V_{peak} = \sqrt{2} \cdot V \qquad I_{peak} = \sqrt{2} \cdot I$$

the **power** delivered to a resistor is:

$$p_R(t) = V \cdot I - V \cdot I \cdot \cos(2 \cdot \omega \cdot t)$$

with an average (**Real Power**) value of:

$$P_{R(AC)} = Avg[p_R(t)] = V \cdot I$$

RMS Magnitudes & Resistor Power

The result:

$$P_{R(AC)} = V \cdot I$$

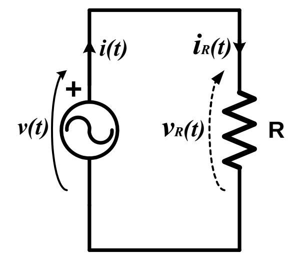
is similar to the DC formula for power:

$$P_{R(DC)} = V_{DC} \cdot I_{DC}$$

which provides the motivation for expressing the AC waveforms in terms of their **RMS** (effective) magnitudes instead of their peak magnitudes.

$$v(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

$$i(t) = \sqrt{2} \cdot I \cdot \sin(\omega \cdot t + \phi)$$



AC Power – General Case

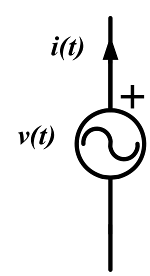
As previously stated, the **general expression** for the **power** produced by an AC source is:

$$p(t) = v(t) \cdot i(t)$$

$$= V_{peak} \cdot I_{peak} \cdot \sin(\omega \cdot t + \phi) \cdot \sin(\omega \cdot t + \delta)$$

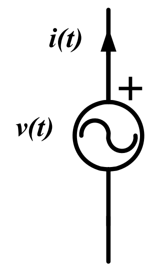
where: $v(t) = V_{peak} \cdot \sin(\omega \cdot t + \phi)$

$i(t) = I_{peak} \cdot \sin(\omega \cdot t + \delta)$



AC Power – General Case

If the voltages & currents are expressed in terms of their **RMS magnitudes**, the power expression becomes:



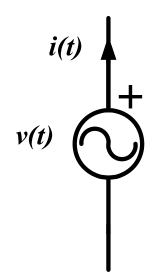
$$p(t) = v(t) \cdot i(t)$$
$$= \sqrt{2} \cdot V \cdot \sqrt{2} \cdot I \cdot \sin(\omega \cdot t + \phi) \cdot \sin(\omega \cdot t + \delta)$$
$$= 2 \cdot V \cdot I \cdot \sin(\omega \cdot t + \phi) \cdot \sin(\omega \cdot t + \delta)$$

which may be modified using several trigonometric identities into the following form:

$$p(t) = V \cdot I \cdot \cos(\phi - \delta)$$
$$- V \cdot I \cdot \cos(\phi - \delta) \cdot \cos(2 \cdot \omega \cdot t)$$
$$+ V \cdot I \cdot \sin(\phi - \delta) \cdot \sin(2 \cdot \omega \cdot t)$$

AC Power – General Case

The modified power expression is often simplified by defining a new variable, $\boldsymbol{\theta}$, where:



$$\theta = \phi - \delta$$

and substituting it into the equation, resulting in the **general power expression**:

$$p(t) = V \cdot I \cdot \cos(\theta)$$
$$- V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t)$$
$$+ V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t)$$

AC Power – General Case

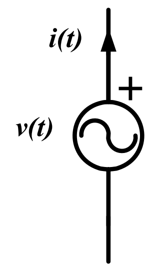
The angle $\boldsymbol{\theta}$, is defined by the difference between the phase angles of the voltage and current,

$$\theta = \angle\tilde{V} - \angle\tilde{I} = \phi - \delta$$

such than: $v(t) = V_{peak} \cdot \sin(\omega \cdot t + \phi)$

$$i(t) = I_{peak} \cdot \sin(\omega \cdot t + \delta)$$

The angle $\boldsymbol{\theta}$ is often referred to as the **power angle**:

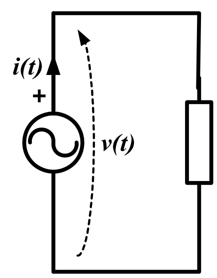


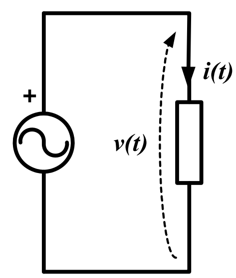
AC Power – General Case

This general expression defines the instantaneous **power produced by an AC source**.

$$p(t) = V \cdot I \cdot \cos(\theta) - V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t) + V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t)$$

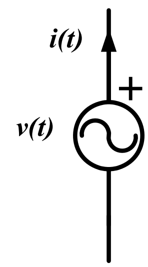
Likewise, if the source is connected across a load that may have resistive, capacitive, and/or inductive components, then the solution also defines the instantaneous **power consumed by the AC supplied load**.





AC Power – General Case

The **general expression** for **AC power** has three terms:

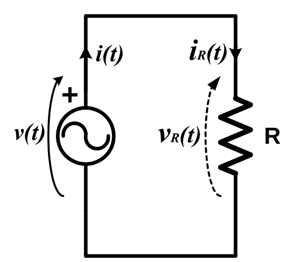


$$p(t) = V \cdot I \cdot \cos(\theta)$$
$$- V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t)$$
$$+ V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t)$$

To help clarify the relevance of each term, let's first look at the power waveforms that result from the source being used to supply either a purely resistive, purely inductive, or purely capacitive load.

AC Power and Resistors

If an AC source is connected to a **resistive** load,



$$v_R(t) = \sqrt{2} \cdot V_R \cdot \sin(\omega \cdot t + \phi)$$

then the resistor current will be:

$$i_R(t) = \sqrt{2} \cdot \frac{V_R}{R} \cdot \sin(\omega \cdot t + \phi)$$

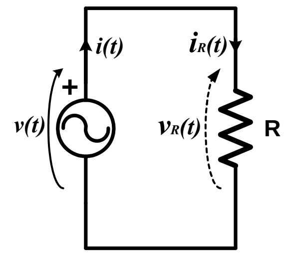
and the **power consumed by the resistor** will be:

$$p_R(t) = V_R \cdot I_R - V_R \cdot I_R \cdot \cos(2 \cdot \omega \cdot t)$$

AC Power and Resistors

The resultant power waveform has two terms:

$$p_R(t) = \boxed{V_R \cdot I_R} - \boxed{V_R \cdot I_R \cdot \cos(2 \cdot \omega \cdot t)}$$



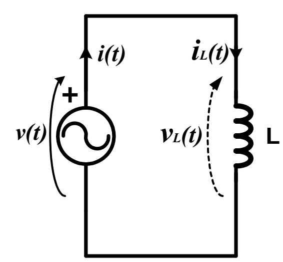
- the first of which is a **constant** that provides the **average** power supplied to the resistor, which is defined to be **Real Power**, $\boldsymbol{P_R}$, and
- the second of which is a **purely sinusoidal** term that has a **zero average** value and varies at 2x the frequency of the source voltage.

$$P_R = V_R \cdot I_R \quad \text{Watts}$$

AC Power and Inductors

If the AC source is connected to an inductive load,

$$v_L(t) = \sqrt{2} \cdot V_L \cdot \sin(\omega \cdot t + \phi)$$



then the inductor current will be:

$$i_L(t) = \sqrt{2} \cdot \frac{V_L}{\omega \cdot L} \cdot \sin(\omega \cdot t + \phi° - 90°)$$

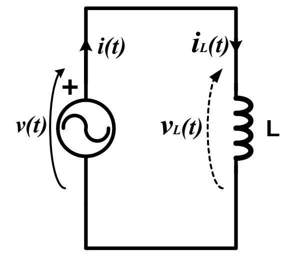
and the power consumed by the inductor will be:

$$p_L(t) = V_L \cdot I_L \cdot \sin(2 \cdot \omega \cdot t)$$

AC Power and Inductors

The resultant power waveform has only one term:

$$p_L(t) = \boxed{V_L \cdot I_L \cdot \sin(2 \cdot \omega \cdot t)}$$



which is a **purely sinusoidal** term that has a **zero average** value and varies at twice (2x) the frequency of the source voltage.

Since the power waveform has a zero-average value, the inductor consumes zero real power:

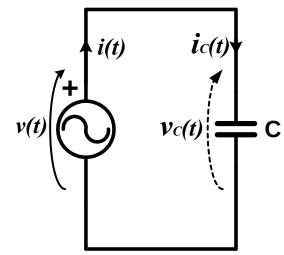
$$P_L = 0 \quad \text{Watts}$$

but power is flowing into and out of the inductor.

AC Power and Capacitors

If the AC source is connected to an capacitor load,

$$v_C(t) = \sqrt{2} \cdot V_C \cdot \sin(\omega \cdot t + \phi)$$



then the capacitor current will be:

$$i_C(t) = \sqrt{2} \cdot V_C \cdot \omega \cdot C \cdot \sin(\omega \cdot t + \phi° + 90°)$$

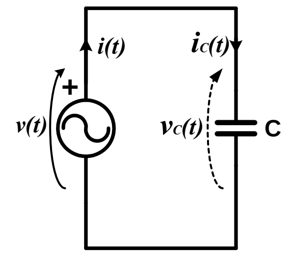
and the power consumed by the capacitor will be:

$$p_C(t) = -V_C \cdot I_C \cdot \sin(2 \cdot \omega \cdot t)$$

AC Power and Capacitors

The resultant power waveform has only one term:

$$p_C(t) = \boxed{-V_C \cdot I_C \cdot \sin(2 \cdot \omega \cdot t)}$$



which is a **purely sinusoidal** term that has a **zero average** value and varies at twice (2x) the frequency of the source voltage.

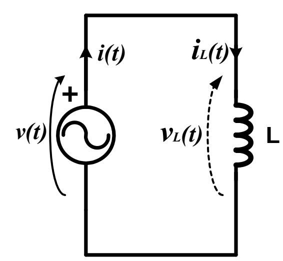
Since the power waveform has a zero-average value, the capacitor consumes zero real power:

$$P_C = 0 \quad \text{Watts}$$

but power <u>is</u> flowing into and out of the capacitor.

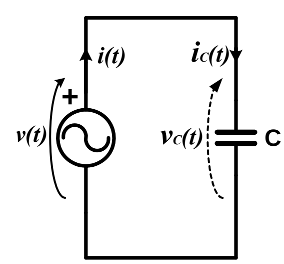
AC Power and Reactive Loads

Although the (average) **real power consumed by both inductors and capacitors is zero**, there <u>is power flowing in and out of these elements</u> when supplied by an AC source:



$$p_L(t) = V_L \cdot I_L \cdot \sin(2 \cdot \omega \cdot t)$$

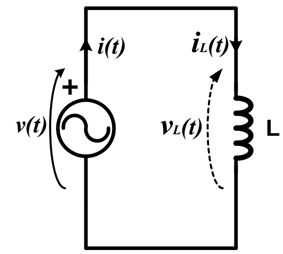
$$p_C(t) = -V_C \cdot I_C \cdot \sin(2 \cdot \omega \cdot t)$$

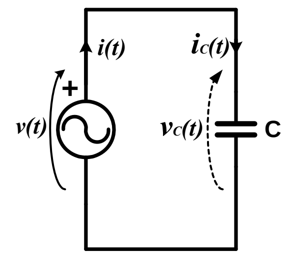


The term **Reactive Power** is used to characterize the amount of energy that is being temporarily stored and then released by a "***reactive load***" (capacitive or inductive).

Reactive Power

Reactive Power (***Q***) is defined as the magnitude of the power that is flowing into and out of a reactive load when supplied by an AC source.





Thus, given: $$p_L(t) = V_L \cdot I_L \cdot \sin(2 \cdot \omega \cdot t)$$

$$p_C(t) = -V_C \cdot I_C \cdot \sin(2 \cdot \omega \cdot t)$$

the reactive power for the inductive and capacitive loads can be defined as:

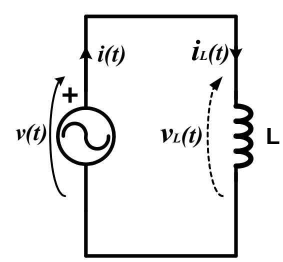
$$Q_L = +V_L \cdot I_L \quad Vars$$

$$Q_C = -V_C \cdot I_C \quad Vars$$

Reactive Power

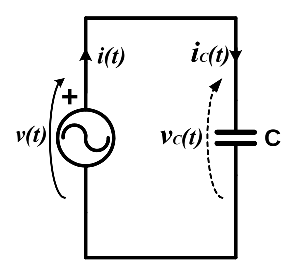
Reactive Power is given the unit of "***Vars***", which stands for:

"***Volt-Amps-Reactive***".



$$Q_L = +V_L \cdot I_L \quad Vars$$

$$Q_C = -V_C \cdot I_C \quad Vars$$

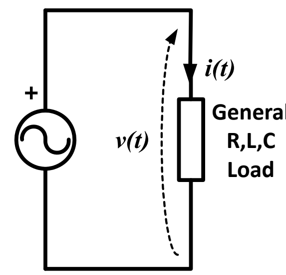


Note that the <u>reactive power for an inductor is **positive**</u> while the <u>reactive power for a capacitor is **negative**</u>.

Thus, it is often stated that an inductor "consumes" reactive power while a capacitor "produces" reactive power.

AC Power – General Case

The previous results can be used to define the relevance of the **three terms** that appear in the **general AC power expression**:



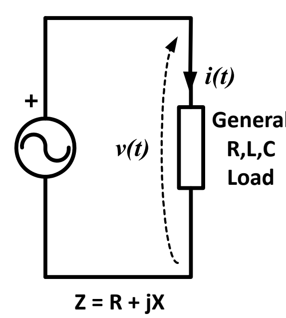
$$p(t) = \boxed{V \cdot I \cdot \cos(\theta)}$$
$$- V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t)$$
$$+ V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t)$$

The **first term** is a **constant** that provides the **average** or **Real Power** that is consumed by the resistive portion of the load:

$$P = V \cdot I \cdot \cos(\theta)$$

AC Power – General Case

The previous results can be used to define the relevance of the **three terms** that appear in the **general AC power expression**:



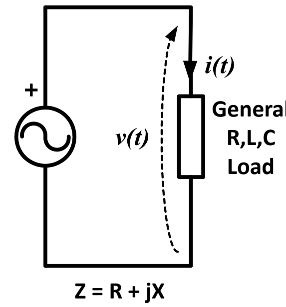
$$p(t) = V \cdot I \cdot \cos(\theta)$$
$$\boxed{- V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t)}$$
$$+ V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t)$$

The **second term** is a sinusoidal term that varies at 2x the source frequency and provides the fluctuation in the power being supplied to the resistive portion of the load.

AC Power – General Case

The previous results can be used to define the relevance of the **three terms** that appear in the **general AC power expression**:

$$p(t) = V \cdot I \cdot \cos(\theta)$$
$$- V \cdot I \cdot \cos(\theta) \cdot \cos(2 \cdot \omega \cdot t)$$
$$\boxed{+ V \cdot I \cdot \sin(\theta) \cdot \sin(2 \cdot \omega \cdot t)}$$



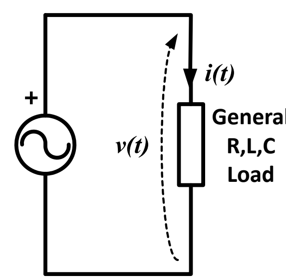
The **third term** is also purely sinusoidal, the magnitude of which provides the **Reactive Power** "consumed" by the reactive portion of the load.

$$Q = V \cdot I \cdot \sin(\theta) \quad Vars$$

AC Power in Combination Circuits

Note that, if the load has **both** a resistive and a capacitive or an inductive component, then the **power angle *θ*** will fall somewhere in the range:

-90°≤ *θ* ≤+90°



resulting in the existence of all three terms in the general power expression.

Thus, there will be **Real** and **Reactive Powers** flowing into the load, as defined by:

$$P = V \cdot I \cdot \cos(\theta)$$
$$Q = V \cdot I \cdot \sin(\theta)$$

Phasor Representation of Sine Waves

A **phasor** is a representation of a **sinusoidal waveform** whose magnitude, phase and frequency are constant.

Phasors reduce the dependency of these parameters to three independent factors, thus allowing for the simplification of certain types of calculations.

It turns out that, for **steady-state AC circuits**, the time dependency of the sine-waves can be factored out, reducing the linear differential equations required for their solution to a simpler set of algebraic equations.

Phasors and AC Voltages

The **sinusoidal voltage**:

$$v(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

may be defined in the form of a **phasor voltage**:

$$\tilde{V} = Ve^{j\phi} = V\angle\phi$$

where it is expressed as a complex number in "***polar***" form, with **RMS magnitude** $\boldsymbol{V}$ and **phase angle** $\boldsymbol{\phi}$.

(Note – although phasor value may be expressed in terms of "***peak***" magnitudes, **RMS voltage magnitudes** are typically used in cases where "***power***" is of primary interest, and thus will be utilized in this course unless specifically stated otherwise.)

Phasors and AC Voltages

$$v(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

$$\tilde{V} = Ve^{j\phi} = V\angle\phi$$

The **phasor voltage** defined above is shown both as a **complex exponential** $Ve^{j\phi}$ and as a **complex number in polar form** $V\angle\phi$

Phasors are typically presented in most "circuits" textbooks as complex numbers expressed in "polar" form, but some calculators do not accept this format, thus requiring the use of complex exponentials.

Phasors and AC Voltages

$$v(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

$$\tilde{V} = Ve^{j\phi} = V\angle\phi$$

Note that although $e^{j\phi} = 1\angle\phi$, mathematically the expression $e^{j\phi}$ is technically only valid if the angle $\boldsymbol{\phi}$ is defined in **radians**.

In practice, $\boldsymbol{\phi}$ <u>is often expressed in degrees</u>, especially when relating to phasors. Some calculators will accept complex exponentials with their angles expressed either in radians or in degrees; other calculators require the angle of the complex exponential to be entered in radians.

Phasors and AC Currents

The **sinusoidal current**:

$$i(t) = \sqrt{2} \cdot I \cdot \sin(\omega \cdot t + \delta)$$

may also be defined in the form of a **phasor current**:

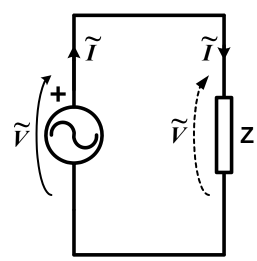
$$\tilde{I} = Ie^{j\delta} = I\angle\delta$$

where it is expressed as a complex number in "***polar***" form, with **RMS magnitude *I*** and **phase angle $\boldsymbol{\delta}$**.

(Note – **RMS current magnitudes** will be also utilized in this course unless specifically stated otherwise.)

Impedance

An **impedance** is a general expression that is used to define the ratio of the phasor value of the voltage across a load compared to the phasor value of the current flowing through the load.



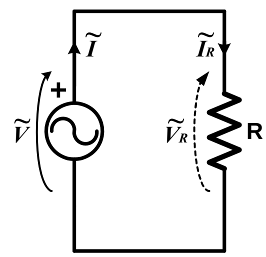
$$Z = \frac{\tilde{V}}{\tilde{I}}$$

Based on this definition, the **impedance *Z*** can be defined in terms of the voltage and current as:

$$Z = \frac{\tilde{V}}{\tilde{I}} = \frac{V\angle\phi}{I\angle\delta} = \frac{V}{I}\angle(\phi - \delta) = \frac{V}{I}\angle\theta = |Z|\angle\theta$$

Impedance of a Resistor

Since **Ohm's Law** holds true for resistors when supplied with any voltage source (including those expressed as phasors), the **impedance of a resistor** is equal to it's **resistance**:

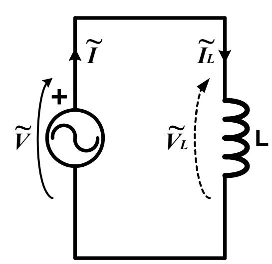


$$Z_R = \frac{\tilde{V}_R}{\tilde{I}_R} = R$$

which is a **purely real number**.

Impedance of an Inductor

Given an **inductor** with voltage $v_L(t)$, the current flowing through the inductor can be defined in terms of the inductor's voltage and inductance:



$$v_L(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

$$i_L(t) = \sqrt{2} \cdot \frac{V}{\omega \cdot L} \cdot \sin(\omega \cdot t + \phi° - 90°)$$

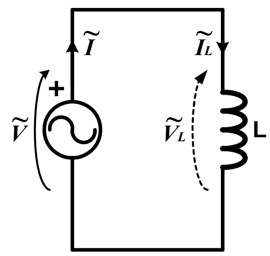
When expressed as **phasors**, the inductor's voltage and current are:

$$\tilde{V}_L = V\angle\phi \qquad \tilde{I}_L = \frac{V}{\omega \cdot L}\angle\phi - 90°$$

Impedance of an Inductor

Based on the inductor's **phasor voltage and current**:

$$\tilde{V}_L = V\angle\phi \qquad \tilde{I}_L = \frac{V}{\omega\cdot L}\angle\phi - 90°$$



the inductor's **impedance** can be defined as:

$$Z_L = \frac{\tilde{V}_L}{\tilde{I}_L} = \frac{V\angle\phi}{\left(\frac{V}{\omega\cdot L}\right)\angle\phi - 90°} = (\omega\cdot L)\angle + 90°$$

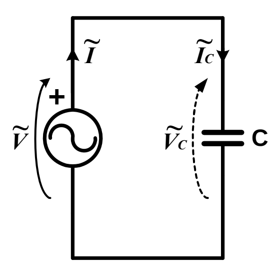
which is typically expressed as a **purely imaginary complex number** in **rectangular form**:

$$Z_L = (\omega\cdot L)\angle + 90° = 0 + j\omega\cdot L = \boxed{j\omega\cdot L}$$

Note that the **impedance of an inductor** is a **positive, imaginary number**

Impedance of a Capacitor

Given a **capacitor** with voltage $\boldsymbol{v_C(t)}$, the current flowing through the capacitor can be defined in terms of the capacitor's voltage and capacitance:



$$v_C(t) = \sqrt{2}\cdot V\cdot\sin(\omega\cdot t + \phi)$$

$$i_C(t) = \sqrt{2}\cdot V\cdot\omega\cdot C\cdot\sin(\omega\cdot t + \phi° + 90°)$$

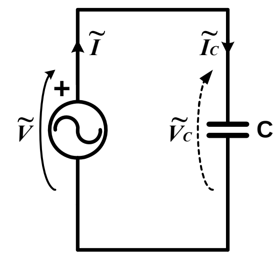
When expressed as **phasors**, the capacitor's voltage and current are:

$$\tilde{V}_C = V\angle\phi \qquad \tilde{I}_C = V\cdot\omega\cdot C\angle\phi + 90°$$

Impedance of a Capacitor

Based on the capacitor's **phasor voltage and current**:

$$\tilde{V} = V\angle\phi \qquad \tilde{I} = V\cdot\omega\cdot L\angle\phi + 90^\circ$$



the capacitor's **impedance** can be defined as :

$$Z_C = \frac{\tilde{V}_C}{\tilde{I}_C} = \frac{V\angle\phi}{V\cdot\omega\cdot C\angle\phi + 90^\circ} = \frac{1}{\omega\cdot C}\angle -90^\circ$$

which is typically expressed as a **purely imaginary complex number** in **rectangular form**:

$$Z_C = \frac{1}{\omega\cdot C}\angle -90^\circ = 0 - j\frac{1}{\omega\cdot C} = \boxed{-j\frac{1}{\omega\cdot C}}$$

Note that the **impedance of a capacitor** is a **positive**, **imaginary number**

Reactance

Reactance defines the manner in which capacitive and inductive loads react to a steady-state sinusoidal voltage.

The **reactance** of a load is equal to the imaginary value of the load's impedance.

Therefore:

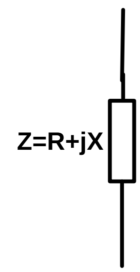
- the reactance of a **resistor** is $X_R = 0\ \Omega$
- the reactance of an **inductor** is $X_L = \omega\cdot L\ \Omega$
- the reactance of a **capacitor** is $X_C = \frac{-1}{\omega\cdot C}\ \Omega$

Complex Impedances

A **complex impedance, *Z*,** can have both resistive and reactive (inductive or capacitive) components, and may be expressed in the form:

$$Z = R + jX$$

where: R is the resistive component of the load, and
X is the reactive component of the load.



Note – the impedance of a resistor is $Z_R = R$

– the impedance of an inductor is $Z_L = j(\omega \cdot L)$

– the impedance of a capacitor is $Z_C = -j\left(\frac{1}{\omega \cdot C}\right)$

Phasor Analysis of AC Circuits

$$\tilde{V} = V\angle\phi$$

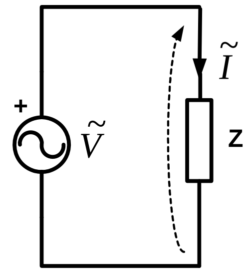
$$\tilde{I} = I\angle\delta$$

When all of the voltages and currents within a **steady-state AC circuit** are expressed as **phasors** and all of the circuit elements are defined by their **impedance** values, the circuit's operation may be solved by a set of algebraic equations that are based on the Ohm's Law equation:

$$\tilde{V} = \tilde{I} \cdot Z$$

Phasor Analysis of AC Circuits

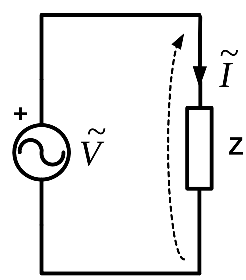
If a voltage source having the **phasor** value $\tilde{V}$ is applied across the complex impedance *Z*, then the **phasor** value of the current $\tilde{I}$ may be solved by applying Ohm's Law:



$$\tilde{I} = \frac{\tilde{V}}{Z} = \frac{\tilde{V}}{R + jX} = I\angle\delta$$

Phasor Analysis of AC Circuits

Similarly, given the **phasor voltage and current** supplied to an **impedance *Z***:



$$\tilde{V} = V\angle\phi \qquad \tilde{I} = I\angle\delta$$

the **impedance** may be defined in terms of voltage and current as:

$$Z = \frac{\tilde{V}}{\tilde{I}} = \frac{V\angle\phi}{I\angle\delta} = \frac{V}{I}\angle(\phi - \delta) = \frac{V}{I}\angle\theta = |Z|\angle\theta$$

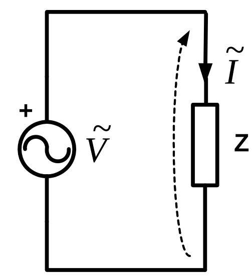
where ***Z*** is a **complex number** expressed polar-form.

Phasor Analysis of AC Circuits

Thus, given:

$$\tilde{V} = V\angle\phi \qquad \tilde{I} = I\angle\delta$$

the **impedance magnitude** is defined by Ohm's Law and the **impedance angle** is the difference between the voltage and current phase angles.

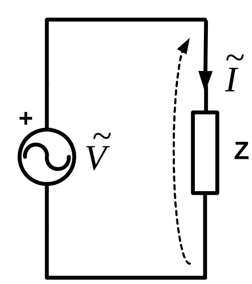


$$Z = |Z|\angle\theta \qquad ==> \qquad |Z| = \frac{V}{I} \qquad \theta = \phi - \delta$$

Note that the impedance angle $\boldsymbol{\theta}$ is the same as the previously defined "***power angle***" $\boldsymbol{\theta}$ that was defined during the AC power analysis.

Phasor Analysis of AC Circuits

Note that the following formulas may be used to **convert an impedance** between rectangular form and polar form:



$$Z = |Z|\angle\theta \qquad ==> \qquad Z = R + jX$$

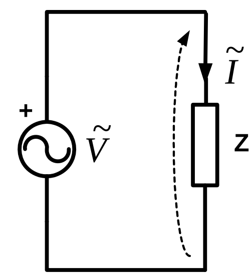
$$\mathrm{R} = |Z|\cdot\cos(\theta) \qquad X = |Z|\cdot\sin(\theta)$$

$$Z = R + jX \qquad ==> \qquad Z = |Z|\angle\theta$$

$$|Z| = \sqrt{R^2 + X^2} \qquad \theta = \tan^{-1}\frac{X}{R}$$

Complex Power

The term **Complex Power** is used to characterize both the **Real Power** and the **Reactive Power** that an AC source is producing or that a complex load impedance (with a resistive component and/or an inductive or capacitive reactive component) is consuming**.**



$$Z = R + jX$$

Complex Power (***S***) is a complex number and is defined by:

$$S = P + j\,Q$$

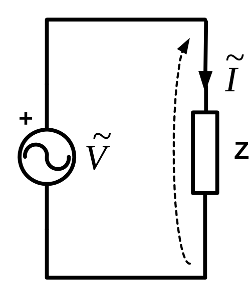
where: ***P*** is **Real Power**, and
Q is **Reactive Power**.

Complex Power

Complex Power (***S***):

$$S = P + j\,Q$$

may be solved directly from a circuit element's **phasor voltage** and **phasor current** as:



$$S = P + jQ = \tilde{V} \cdot \tilde{I}^* = (V\angle\phi)\cdot(I\angle-\delta)$$

$$= V \cdot I \angle(\phi-\delta) = V \cdot I \angle\theta$$

$$= \boxed{V \cdot I \cdot cos\,\theta + j\,V \cdot I \cdot sin\,\theta}$$

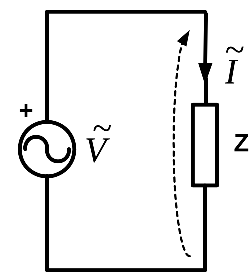
the **real portion** of which relates to **Real Power** and the **imaginary portion** of which relates to **Reactive Power**.

Complex Conjugate

Note that $\tilde{I}^*$ is the **complex conjugate** of $\tilde{I}$, and is defined as:

$$\tilde{I}^* = (I\angle\delta)^* = (I\angle-\delta)$$

The **complex conjugate** of a complex number expressed in **polar form** has the **same magnitude as the original number** but the **angle is negated**.



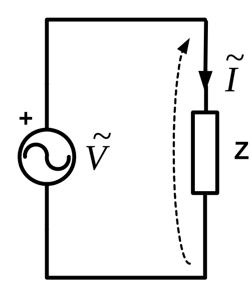
Apparent Power

Apparent Power (|***S***|) is defined to be the **magnitude of complex power**:

$$|S| = V \cdot I = \sqrt{P^2 + Q^2}$$

Note that **apparent power** is often specified as one of the "***ratings***" of a machine, such that:

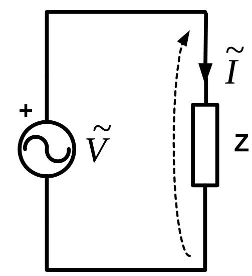
$$|S|_{rated} = V_{rated} \cdot I_{rated}$$



Power Factor

Power Factor (***pf***) provides a measure of the portion of the apparent power that relates to real power:

$$pf = \frac{P}{|S|}$$



Thus, **power factor** may be defined as:

$$pf = \frac{P}{|S|} = \frac{V \cdot I \cdot \cos\theta}{V \cdot I} = \boxed{\cos\theta}$$

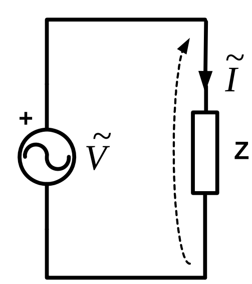
Note that **power factor** is often specified as having a **leading** or **lagging** characteristic, which is based on the angle relationship between the phasor voltage and the phasor current

Power Factor

A **leading** power factor exists when the current waveform is "leading" the voltage, which occurs with a **capacitive** load impedance and results in a **negative** angle difference for ***θ***:

$$\theta = \phi - \delta$$

$$-90° \leq \theta < 0°$$



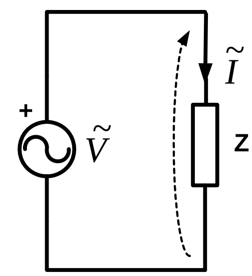
A **lagging** power factor exists when the current waveform is "lagging" the voltage, which occurs with a **inductive** load impedance and results in a **positive** angle difference for ***θ***:

$$\theta = \phi - \delta$$

$$0° < \theta \leq +90°$$

Power Factor

Note that the **angle θ** for a **purely resistive load** has a **zero** value since the voltage and current waveforms are "***in-phase***", resulting in a power factor that which is neither leading nor lagging.



This is referred to as a **unity power factor** since the value of **power factor** under this condition equals **one**.

$$\phi = \delta$$

$$\theta = \phi - \delta = 0°$$

$$\cos(\theta) = \cos(0°) = 1$$

Summary of Complex Power Equations

Complex Power (S): $S = P + jQ = \tilde{V} \cdot \tilde{I}^*$

Real Power (P): $P = V \cdot I \cdot \cos\theta$

Reactive Power (Q): $Q = V \cdot I \cdot \sin\theta$

Apparent Power (|S|): $|S| = V \cdot I = \sqrt{P^2 + Q^2}$

Power Factor (pf): $pf = \cos\theta$