ECET 2111

Circuits II

Basic Electric Circuits

♦

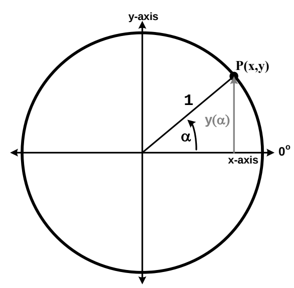
AC Review

Sinusoidally Varying Waveforms

The Sine Function

Given a point **P(x,y)** plotted on a unit circle centered at the origin of the x-y plane, the instantaneous y-value of the point as a function of its angular position **α** on the circle can be used to define the **sine function**, where:

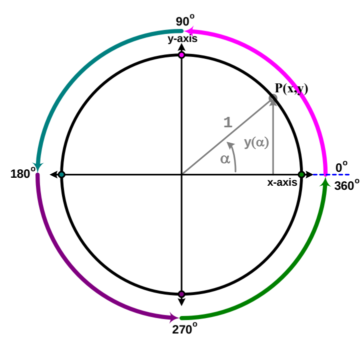
$$y(\alpha) = \sin(\alpha)$$



The Sine Function

The function, **sin(α)**, is a ***periodic function*** that repeats with every **360°** or **2π radian** increase in the angle **α**:

- As **α** varies from **0°→ 360°**, **sin(α)** varies from **0→1→0→ -1→0**, repeating again with every additional **360°** increase in **α**.

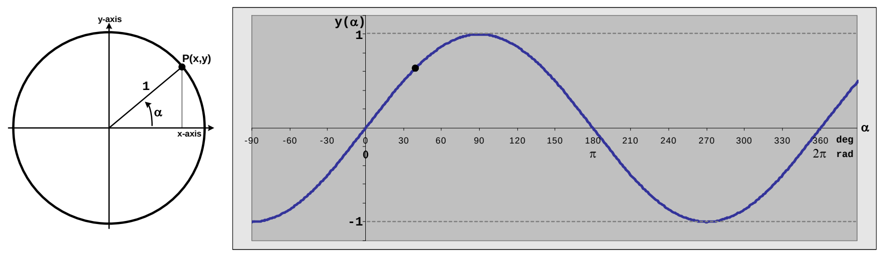


The Sine Function

The function:

$$y(\alpha) = \sin(\alpha)$$

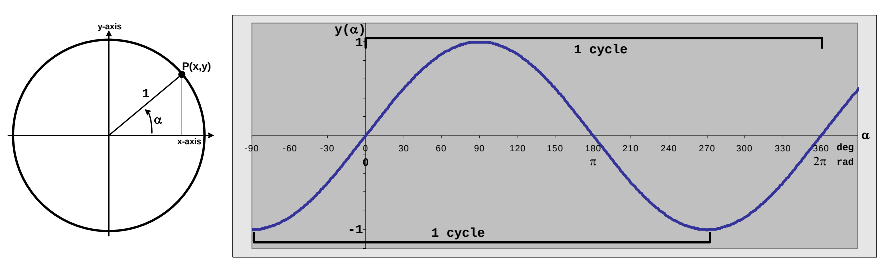
can also be shown by plotting **y(α)** as a function of angle **α**:



The Sine Function

One **cycle** of a periodic waveform is the smallest portion of the waveform that, if repeated continuously, will reproduce the entire waveform.

- The function **y(α)** completes one cycle of variation every time **α** increases by **360°**.

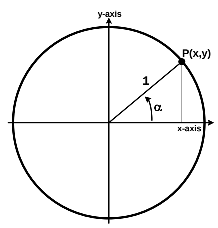


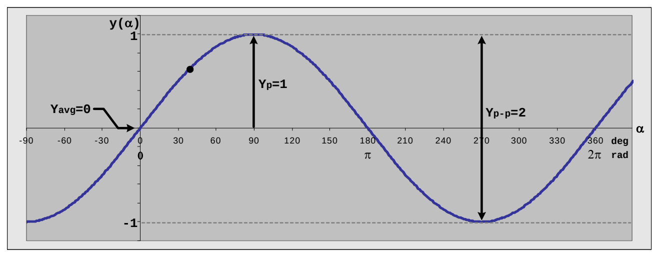
The Sine Function – Amplitude

As shown below, the function:

$$y(\alpha) = \sin(\alpha)$$

- has a **peak magnitude** $Y_p = 1$,
- has a **peak-to-peak magnitude** $Y_{p\text{-}p} = 2$, and
- has an **average** value $Y_{avg} = 0$.





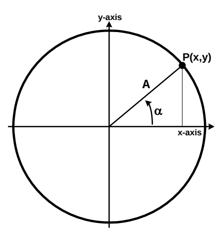
The Sine Function

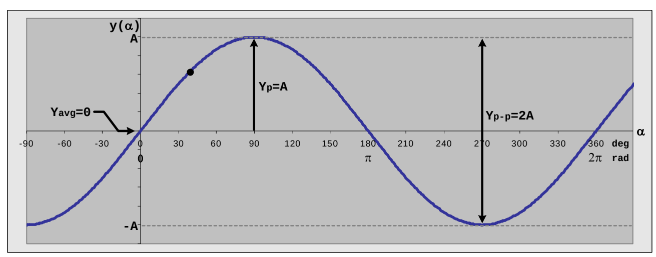
When the sine function is multiplied by a (real) constant ***A***:

$$y(\alpha) = A\sin(\alpha)$$

the **peak magnitude** and the **peak-to-peak magnitude** are both multiplied by ***A***.

(The ***average*** value and the ***repetition interval*** both remain unchanged)





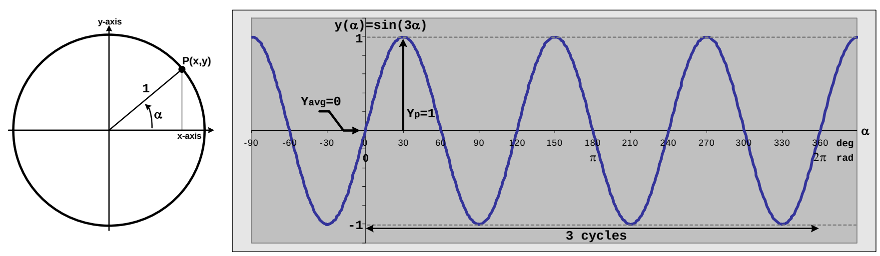
The Sine Function

If the angle-term within the sine function is multiplied by a constant ***B***:

$$y(\alpha) = \sin(B\alpha)$$

then the waveform will repeat ***B*** times within the **360°** interval.

(Note: ***B=3*** as shown in the figure below)



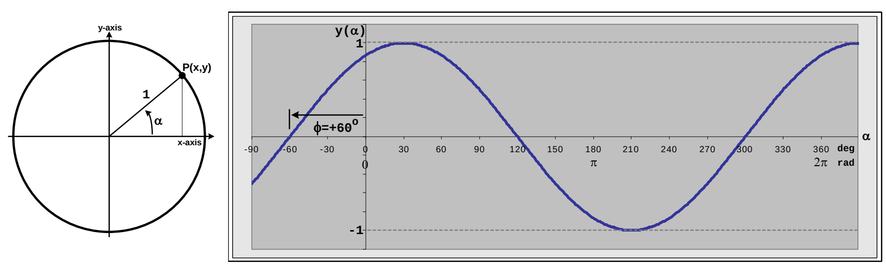
The Sine Function – Phase Shift

If a constant ***ϕ*** is added to the angle-term within the sine function:

$$y(\alpha) = \sin(\alpha + \phi)$$

then the entire waveform will shift to the left or to the right by the angle ***ϕ***.

(Note: ***ϕ = 60°*** as shown in the figure below)



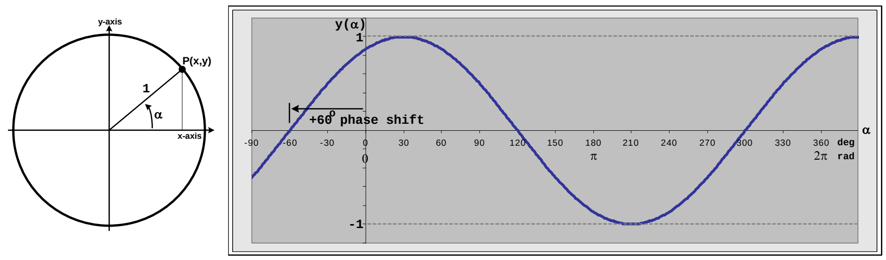
The Sine Function – Phase Shift

The addition of a ***positive*** constant ϕ to the angle-term within the sine function:

$$y(\alpha) = \sin(\alpha + \phi)$$

results in a **phase shift** of the waveform to the ***left*** by the angle ϕ.

(Note: $\phi = 60°$ as shown in the figure below)



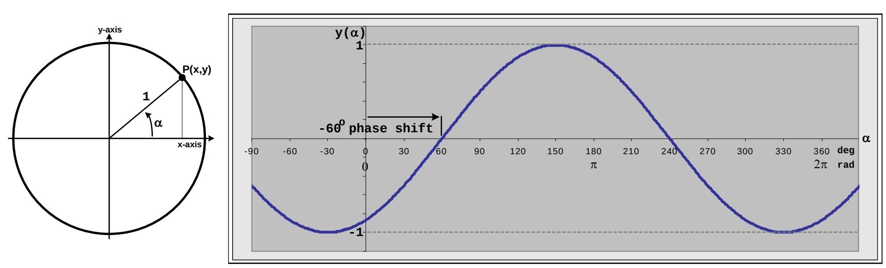
The Sine Function – Phase Shift

The addition of a ***negative*** constant ϕ to the angle-term within the sine function:

$$y(\alpha) = \sin(\alpha + \phi)$$

results in a **phase shift** of the waveform to the ***right*** by the angle ϕ.

(Note: $\phi = -60°$ as shown in the figure below)

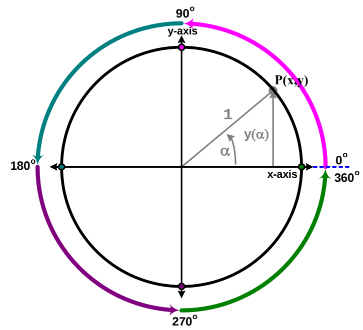


Sine as a Function of Time

Refer back to point **P(x,y)** that is rotated around a unit circle:

Each time **P** rotates one complete revolution around the circle, the function ***y(α)*** progresses through one cycle of its waveform.

What if **P** is continuously rotated around the circle at a constant rate?

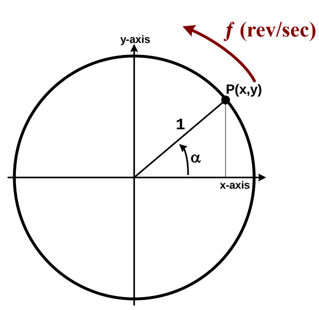


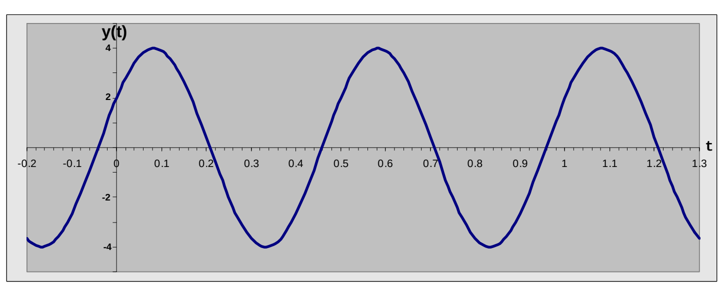
Sine as a Function of Time

If **P** is rotating at a constant rate, ***f***, such that ***f*** is the number of revolutions per second that **P** rotates…

Then the sine function ***y(t)*** will progress through ***f*** cycles of its waveform each second.

(As shown below, ***f*** =2 revolutions per second)



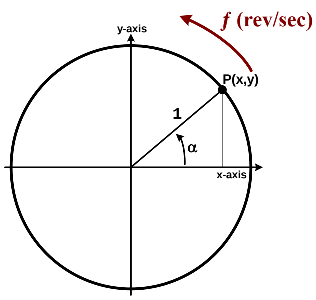


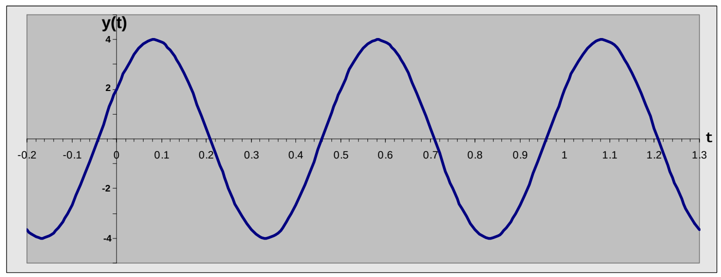
Frequency

The **frequency**, ***f***, of a periodic function is defined as the number of cycles that the function will progress through in one second.

Frequency is assigned the standard unit **Hertz**, where:

Hertz = cycles/sec

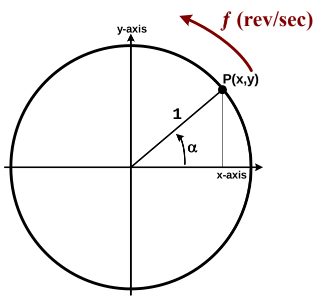


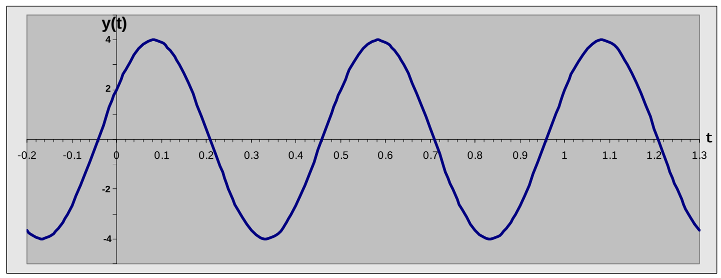


Sine as a Function of Time

If **P** is rotating at a constant rate ***f*** (rev/sec)…

Then the angle ***α*** increases at a rate of $2\pi \cdot f$ (rad/sec) since each rotation relates to an angular increase of 2π radians.



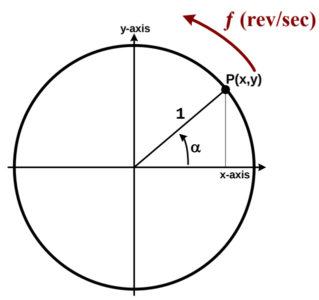


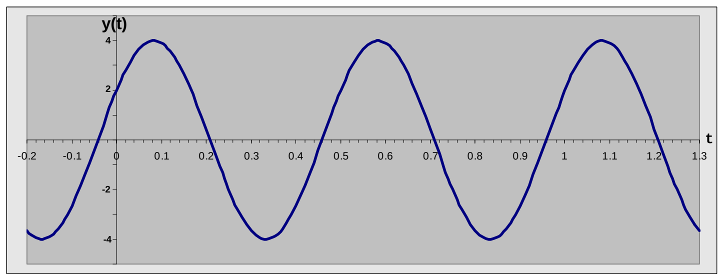
Angular Velocity

The **Angular Velocity** (*ω*) of a sine function is the angular rate at which the sine-angle increases:

$$\omega = 2\pi \cdot f \text{ (rad/sec)}$$

Note – Although angular velocity, *ω*, can be expressed in units of deg/sec, the standard units for *ω* is rad/sec.





Sine as a Function of Time

If ***Angular Velocity*** (*ω*) defines the rate at which the angle of the sine function increases in radians/sec…

$$\omega = 2\pi \cdot f \text{ (rad/sec)}$$

then $\omega \cdot t$ defines a ***radian*** angle if t is expressed in seconds.

And, since the term $\omega \cdot t$ defines an angle, it can be used in place of the angle α in the sine function provided:

$$\alpha = \omega \cdot t \text{ (radians)}$$

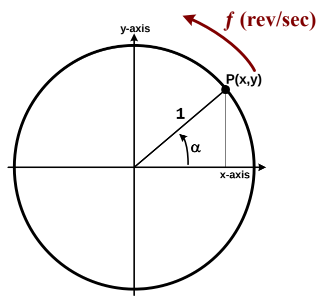
Sine as a Function of Time

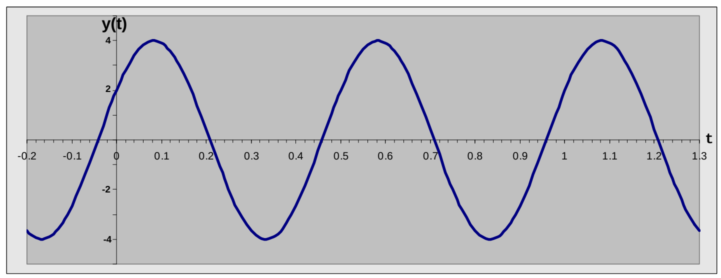
Thus, we can express sine as a function of time, ***y(t)***, such that:

$$y(t) = \sin(\omega \cdot t)$$

where:

$$\omega = 2\pi \cdot f$$

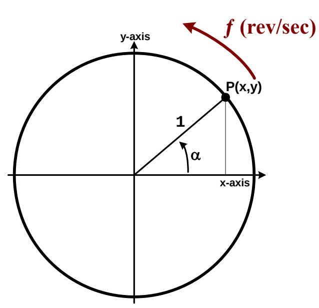


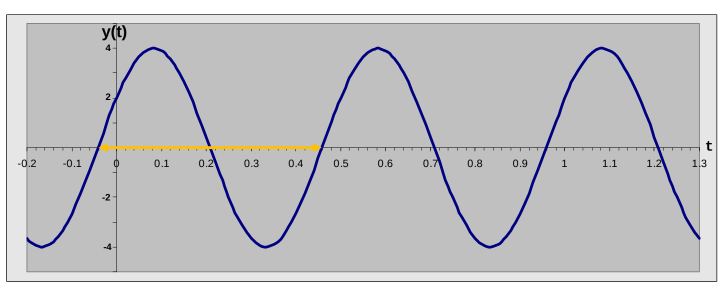


Period

The **<u>period</u>**, ***T***, of a periodic function is the length of time it takes for the function to progress through one cycle of its waveform.

Note – Period is typically defined in standard units of seconds.





Period

Since the function:

$$y(t) = \sin(\omega \cdot t)$$

progresses through ***f*** cycles of its waveform each second, then the **period** of the function (i.e. – the amount of time required to complete each cycle) can be determined from frequency as:

$$T = \frac{1}{f}\left(\frac{1}{Hertz}\right) = \frac{1}{f}\left(\frac{1}{cycles/\sec}\right) = \frac{1}{f}\left(\frac{\sec}{cycle}\right)$$

Period

Thus, given a sine function having frequency ***f***:

$$y(t) = \sin(\omega \cdot t) \qquad \omega = 2\pi \cdot f$$

the **period** of the function can be determined by:

$$T = \frac{1}{f} = \frac{2\pi}{\omega}\left(\frac{\sec}{cycle}\right)$$

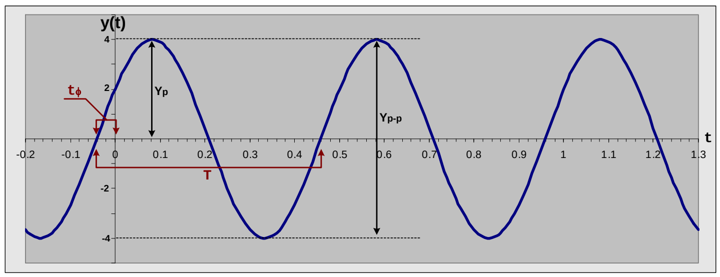
Or, given a periodic waveform having period ***T***, the frequency of the waveform can be determined by:

$$f = \frac{1}{T}\left(\frac{\sec}{cycle}\right)^{-1} = \frac{1}{T}\left(\frac{cycles}{\sec}\right) = \frac{1}{T}(Hertz)$$

Sine as a Function of Time

Given a sinusoidally varying waveform with a peak magnitude Y_p, an angular frequency **ω**, and a phase angle ϕ, the waveform may be expressed as a time function ***y(t)***, where:

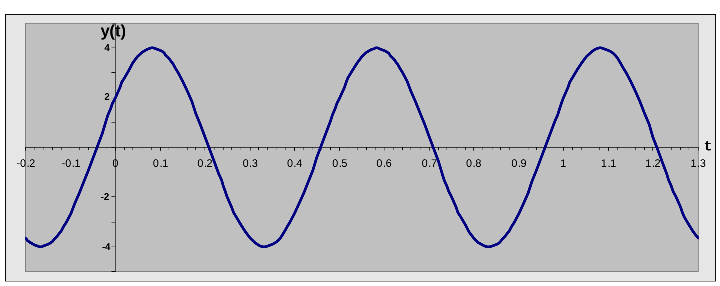
$$y(t) = Y_p \cdot \sin(\omega \cdot t + \phi)$$



Determining Sine from a Time Plot

Given a sinusoidal waveform, ***y(t)***, plotted as a function of time ***t***, determine the following characteristics of the waveform:

Peak Magnitude	Peak-to-Peak Magnitude
Period	Frequency
Angular Frequency	Phase Shift



Determining Sine from a Time Plot

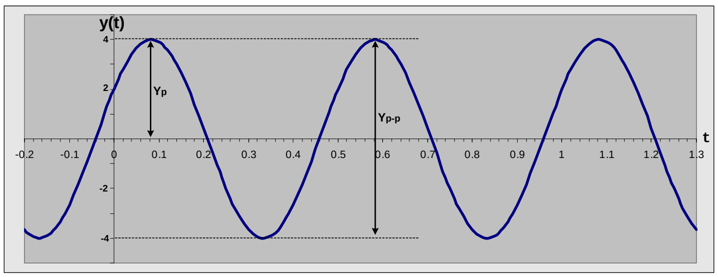
From the plot of ***y(t)*** shown below:

Peak Magnitude

$Y_p = 4$ volts

Peak-to-Peak Magnitude

$Y_{p\text{-}p} = 4 - (-4) = 8$ volts



Determining Sine from a Time Plot

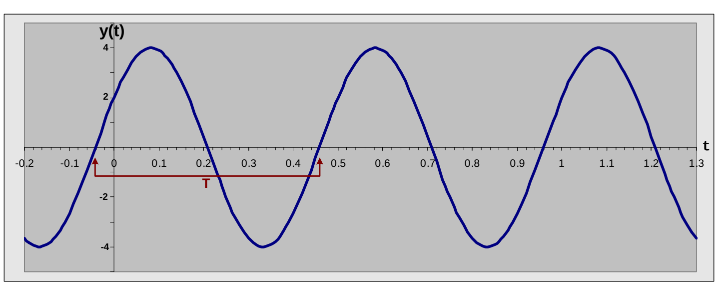
From the plot of ***y(t)*** shown below:

Period

$T = 0.46 - (-0.04) = 0.50$ sec

Frequency

$f = T^{-1} = (0.50)^{-1} = 2$ Hertz (cycles/sec)

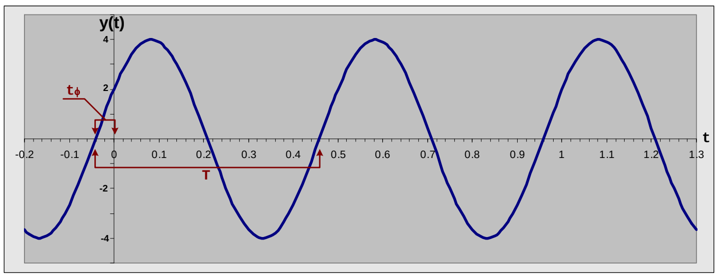


Determining Sine from a Time Plot

Utilizing the value of the frequency ***f***:

Angular Frequency

$\omega = 2\pi \cdot f = 4\pi$ (radians/sec)



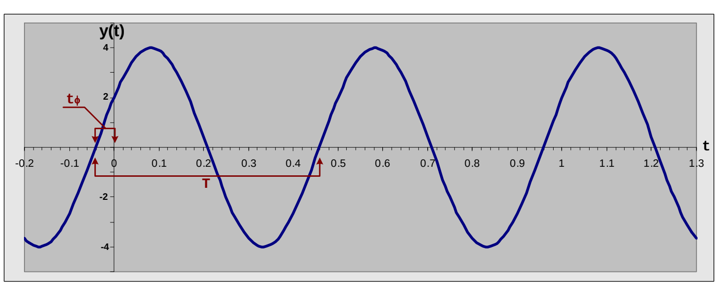
Determining Sine from a Time Plot

From the plot of ***y(t)*** shown below:

Phase Shift

$t_\phi = 0.04$ (seconds), converted to degrees

$$\phi = \frac{t_\phi}{T} \cdot 360° = \frac{0.04}{0.50} \cdot 360° = 28.8°$$



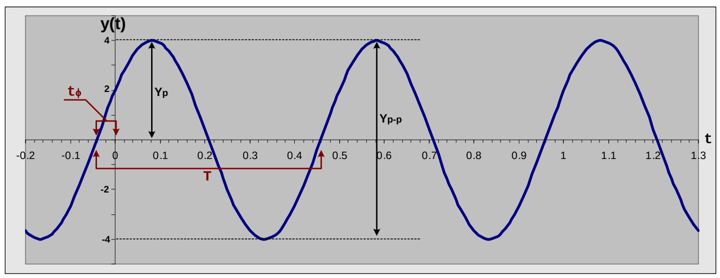
Determining Sine from a Time Plot

Thus, given the plot of the function ***y(t)***, where:

$$y(t) = Y_p \cdot \sin(\omega \cdot t + \phi)\ volts$$

the exact expression for ***y(t)*** is:

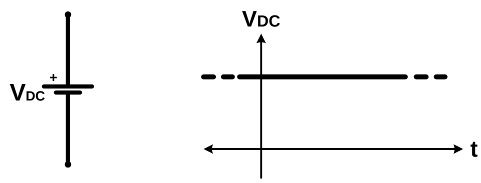
$$y(t) = 4 \cdot \sin(4\pi \cdot t + 28.8°)\ volts$$



AC Circuits

Time-varying Voltages and Currents

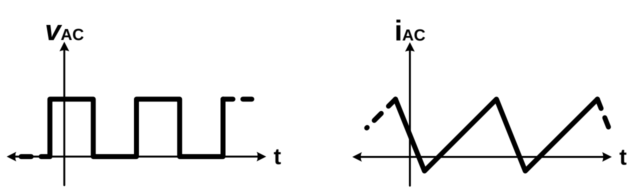
DC voltages and **DC currents**, such as those supplied by an ideal battery, remain constant in time once steady-state conditions are reached.



Time-varying Voltages and Currents

There is another class of voltages and currents, called **AC voltages** and **AC currents**, whose magnitudes vary in a periodic manner as time increases under steady-state conditions.

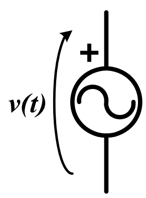
Although the term “**AC**” actually stands for “**Alternating Current**”, it is used to describe both time-varying voltages and currents.

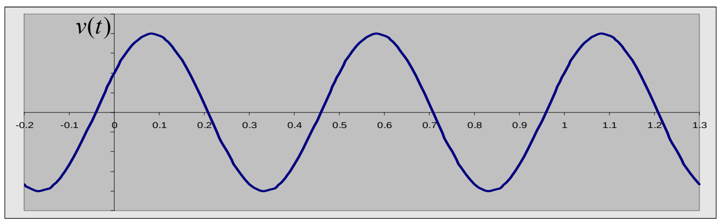


AC Voltage Sources

The most common type of AC voltages (and currents) are those that vary in a **sinusoidal manner**, as shown in the figure below.

Sinusoidally-varying AC voltages are typically created either by rotating machines (generators) or by electronic devices (AC power supplies).

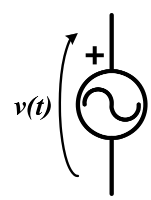


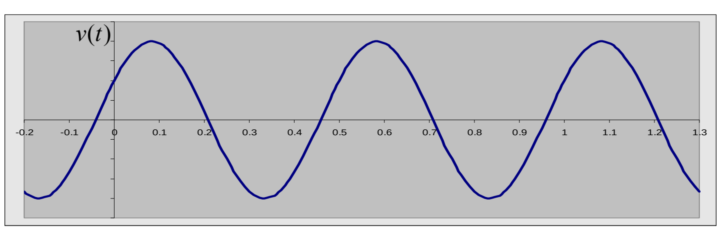


AC Voltage Sources

The "+" sign is used to define the direction of the voltage-rise (potential force) provided by the source as defined by the function ***v(t)***.

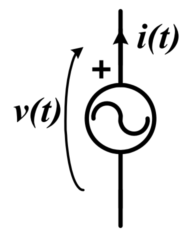
When the function goes "negative", then the force defined by the voltage-rise is in the "negative" or opposite direction.



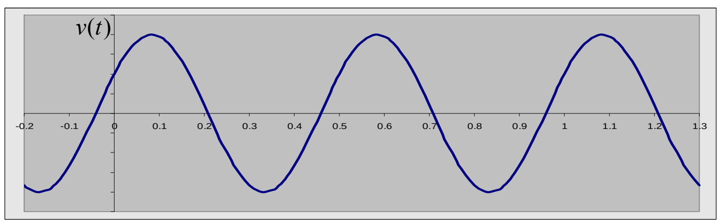


AC Voltage Sources

Since both the magnitude and sign of the voltage potential across the "AC" source vary with time, taking on both positive and negative values, the direction of the resultant current will also vary with time…

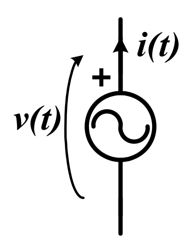


Thus the term → **Alternating Current**



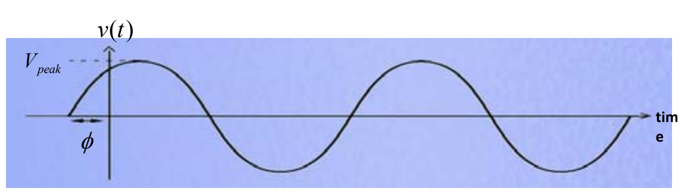
Steady-State AC Voltage Sources

The voltage potential of an AC source may be defined as:



$$v(t) = V_{peak} \cdot \sin(\omega \cdot t + \phi)$$

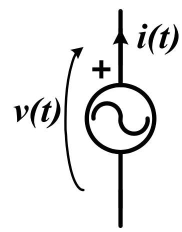
where: V_{peak} is the peak value of the voltage,
ω is the angular frequency ($2\pi f$) of the waveform, and
ϕ is the phase angle of the voltage waveform.



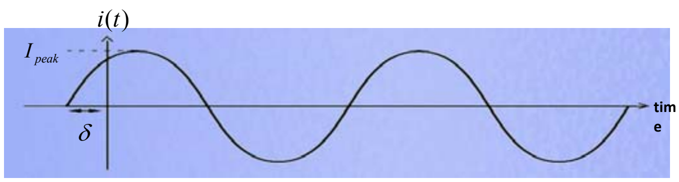
Steady-State AC Voltage Sources

Similarly, the current produced by the AC source may be defined as:

$$i(t) = I_{peak} \cdot \sin(\omega \cdot t + \delta)$$



where: I_{peak} is the peak value of the current,
ω is the angular frequency ($2\pi f$) of the waveform, and
δ is the phase angle of the current waveform.

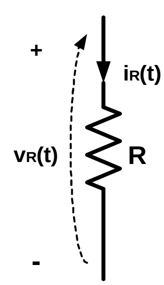


AC Sources and Resistive Loads

<u>**Ohm's Law**</u> defines the voltage/current relationship for a resistive load. This relationship holds true for all types of voltages and currents, including both AC and DC.

→ A time-varying voltage $\boldsymbol{v_R(t)}$ supplied across a resistor will result in a time-varying current $\boldsymbol{i_R(t)}$ flowing through the resistor, such that:

$$i_R(t) = \frac{v_R(t)}{R}$$



AC Sources and Resistive Loads

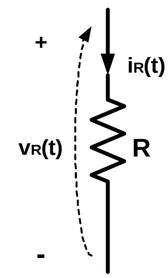
Given a resistor, whose voltage is:

$$v_R(t) = V_{peak} \cdot \sin(\omega \cdot t + \phi)$$

the resultant resistor current will be:

$$i_R(t) = \frac{v_R(t)}{R} = \frac{V_{peak} \cdot \sin(\omega \cdot t + \phi)}{R}$$

$$= \frac{V_{peak}}{R} \cdot \sin(\omega \cdot t + \phi)$$

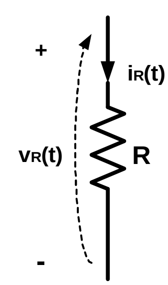


AC Sources and Resistive Loads

Thus, for a resistive load:

$$v_R(t) = V_{peak} \cdot \sin(\omega \cdot t + \phi)$$

$$i_R(t) = \frac{V_{peak}}{R} \cdot \sin(\omega \cdot t + \phi)$$



Note that the voltage and current **magnitudes** follow the Ohm's Law relationship:

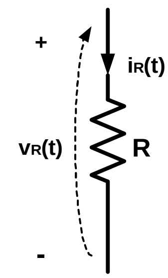
$$I_{peak} = \frac{V_{peak}}{R}$$

AC Sources and Resistive Loads

Thus, for a resistive load:

$$v_R(t) = V_{peak} \cdot \sin(\omega \cdot t + \phi)$$

$$i_R(t) = \frac{V_{peak}}{R} \cdot \sin(\omega \cdot t + \phi)$$



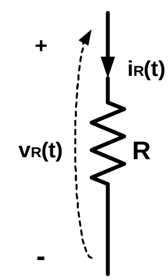
Also note that the **phase angle** of the resistor current is equal to the phase angle of the applied voltage…

There is no phase angle difference between the voltage and current waveforms relating to a purely resistive load.

AC Sources and Resistive Loads

Since Ohm's Law holds true for resistive loads supplied with AC voltages,

$$i_R(t) = \frac{v_R(t)}{R}$$



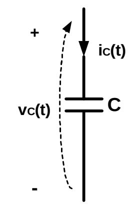
all of the basic circuit theory derived for DC circuits can also be applied to AC resistor circuits:

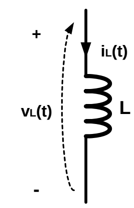
- Series and Parallel Equivalent Resistances
- Kirchhoff's Voltage and Current Laws
- Voltage and Current Dividers
- The Superposition Theorem
- Thevenin's Theorem
- The Maximum Power Transfer Theorem

AC Sources and Reactive Loads

What if the AC source is supplying a load that is purely reactive…

I.e. – either Capacitive or Inductive?





Similar to resistive loads, a sinusoidal (AC) voltage source will cause a sinusoidal (AC) current to flow through both capacitors and inductors.

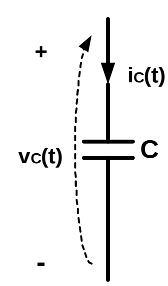
But, their voltage and current waveforms do not follow the linear Ohm's Law relationship. Instead, their voltage and current waveforms are governed by a differential relationship.

AC Sources and Capacitors

For an ideal capacitor, the voltage-current relationship is defined by the following equations:

$$i_C(t) = C \cdot \frac{dv_C(t)}{dt}$$

$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(t)dt = \frac{1}{C}\int_{0}^{t} i_C(t)dt + V_o$$



We may obtain a solution for steady-state AC operation from these relationships.

AC Sources and Capacitors

Given a sinusoidal voltage applied across a capacitor:

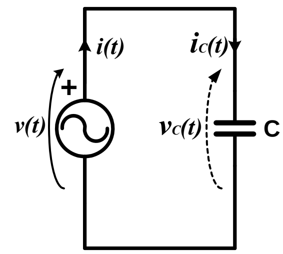
$$v_C(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

the associated capacitor current will be:

$$i_C(t) = \sqrt{2} \cdot V \cdot \omega \cdot C \cdot \cos(\omega \cdot t + \phi°)$$

To allow for direct comparison, the cosine function can be converted to an equivalent sine function using the identity:

$$\cos(x) = \sin(x + 90°)$$



AC Sources and Capacitors

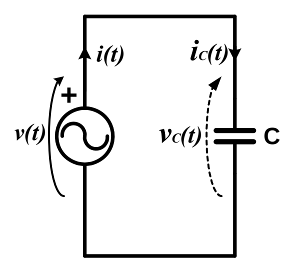
The resultant capacitor voltage and current waveforms, expressed as sine functions, are:

$$v_C(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

$$i_C(t) = \sqrt{2} \cdot V \cdot \omega \cdot C \cdot \sin(\omega \cdot t + \phi° + 90°)$$

Note that:

- The capacitor current is phase-shifted by +90° compared to the capacitor voltage, and
- The voltage and current magnitudes do not follow the linear Ohm's Law relationship that holds true for resistors.



AC Sources and Inductors

Given the sinusoidal voltage applied to an inductor:

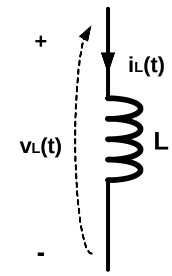
$$v_L(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

the associated current will be:

$$i_L(t) = \sqrt{2} \cdot \frac{V}{\omega \cdot L} \cdot \sin(\omega \cdot t + \phi° - 90°)$$

resulting in a power angle:

$$\theta = \phi - \delta = \phi - (\phi - 90) = +90°$$



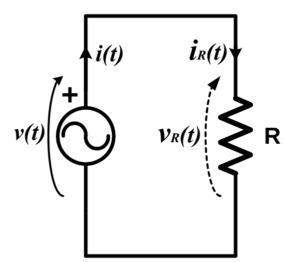
RMS Voltage Magnitude

The effective voltage of a sinusoidal source:

$$V_{effective} = \frac{V_{peak}}{\sqrt{2}}$$

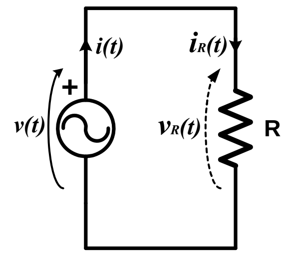
is equal to the **RMS** (***root-mean-squared***) value of the purely sinusoidal voltage, as defined by the function:

$$V_{RMS} = \sqrt{\frac{1}{T} \cdot \int_0^T v^2(t) \cdot dt} = \frac{V_{peak}}{\sqrt{2}}$$



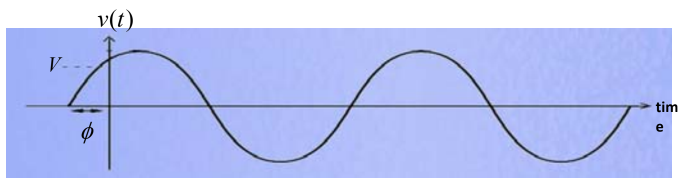
RMS Voltage Magnitude

Thus, the voltage waveform may be expressed in terms of its ***RMS voltage magnitude***:



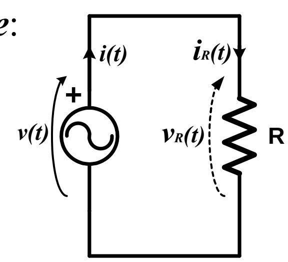
$$v(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

where: $V = \frac{V_{peak}}{\sqrt{2}}$ is the RMS magnitude of the AC voltage.



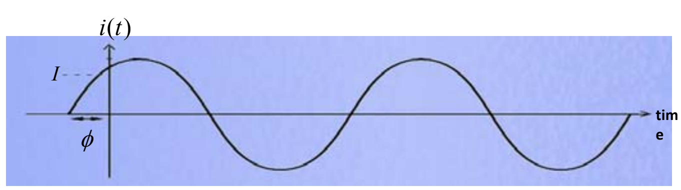
RMS Current Magnitude

Similarly, the current waveform may also be expressed in terms of its ***RMS current magnitude***:



$$i(t) = \sqrt{2} \cdot I \cdot \sin(\omega \cdot t + \phi)$$

where: $I = \frac{I_{peak}}{\sqrt{2}}$ is the RMS magnitude of the AC current.



Phasor Representation of Sine Waves

A **phasor** is a representation of a sine-wave whose magnitude, phase and frequency are constant.

Phasors reduce the dependency of these parameters to three independent factors, thus allowing for the simplification of certain types of calculations.

It turns out that, for steady-state AC circuits, the time dependency of the sine-waves can be factored out, reducing the linear differential equations required for their solution to a simpler set of algebraic equations.

Phasors and AC Voltages

The sinusoidal voltage:

$$v(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

may be defined in the form of a ***phasor voltage***:

$$\tilde{V} = Ve^{j\phi} = V\angle\phi$$

in which the voltage is expressed as a complex number in "***polar***" form, having the RMS magnitude V and the phase angle ϕ.

(Note – although the phasor value may be expressed in terms of "peak" magnitudes, RMS voltage magnitudes will be utilized in this course unless specifically stated otherwise.)

Phasors and AC Voltages

For example, given the sinusoidal voltage:

$$v(t) = 100 \cdot \sin(377 \cdot t + 30°) \quad \text{volts}$$

which may be expressed in terms of its RMS magnitude:

$$v(t) = \sqrt{2} \cdot 70.7 \cdot \sin(377 \cdot t + 30°) \quad \text{volts}$$

The phasor representation of this voltage is:

$$v(t) \quad \Leftrightarrow \quad \tilde{V} = 70.7 e^{j\frac{\pi}{6}} = 70.7\angle 30° \quad \text{volts}$$

such that: $30° = 30° \cdot \dfrac{2\pi \text{ radians}}{360°} = \dfrac{\pi}{6} \text{ radians}$

Phasors and AC Currents

The sinusoidal current:

$$i(t) = \sqrt{2} \cdot I \cdot \sin(\omega \cdot t + \delta)$$

may also be defined in the form of a ***phasor current***:

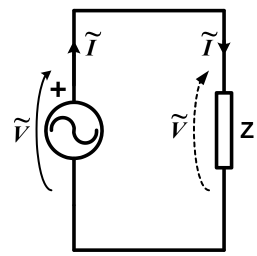
$$\tilde{I} = Ie^{j\delta} = I\angle\delta$$

in which the current is expressed as a complex number in "***polar***" form, having the RMS magnitude I and the phase angle δ.

(Note –RMS current magnitudes will also be utilized in this course unless specifically stated otherwise.)

Impedance

The ***impedance*** of a load provides a measure of the response that the load will have when supplied by a steady-state AC waveform.



Specifically, the ***impedance*** value of a load, ***Z***, is defined as the ratio of the phasor voltage that is applied across the load over the phasor current that flows through the load:

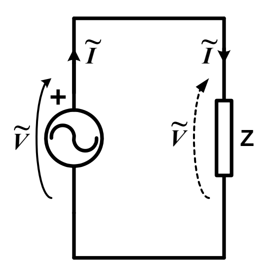
$$Z = \frac{\tilde{V}}{\tilde{I}} \quad (\Omega)$$

the standard units of which are Ohms.

Impedance

Based on the impedance expression:

$$Z = \frac{\tilde{V}}{\tilde{I}}$$



an Ohm's Law type of relationship between the phasor values of the load voltage and current can be defined:

$$\tilde{V} = \tilde{I} \cdot Z$$

which means that any of the DC circuit theory that was derived based on Ohm's Law can also be applied to steady-state AC circuits whose loads are expressed as impedances and whose voltages and currents are expressed by their phasor values.

Impedance

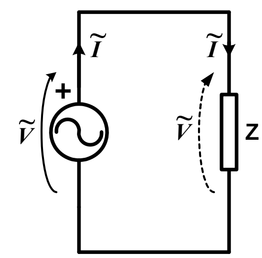
Thus, given the phasor values of the load voltage and current:

$$\tilde{V} = V\angle\phi \qquad \tilde{I} = I\angle\delta$$

the ***impedance***, Z, can be expressed in terms of their phasor values as :

$$Z = |Z|\angle\theta = \frac{\tilde{V}}{\tilde{I}} = \frac{V\angle\phi}{I\angle\delta} = \frac{V}{I}\angle(\phi - \delta)$$

where: $\quad |Z| = \frac{V}{I} \qquad \theta = \phi - \delta$



Note – Impedance is typically expressed as a complex written in "rectangular" form: *Z=R+jX*

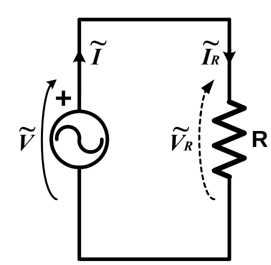
Impedance of a Resistor

Given the voltage across a resistor:

$$v_R(t) = \sqrt{2} \cdot V_R \cdot \sin(\omega \cdot t + \phi)$$

the current flowing through the resistor will be:

$$i_R(t) = \sqrt{2} \cdot \frac{V_R}{R} \cdot \sin(\omega \cdot t + \phi)$$



$$v_R(t) = i_R(t) \cdot R$$

When expressed as phasors, the resistor's voltage and current can be rewritten as:

$$\tilde{V}_R = V_R\angle\phi \qquad \tilde{I}_R = \frac{V_R}{R}\angle\phi$$

Impedance of a Resistor

Based on the values of its phasor voltage and current:

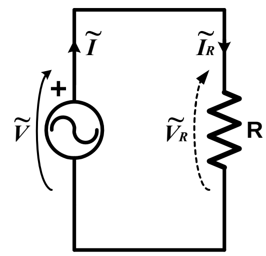
$$\tilde{V}_R = V_R \angle \phi \qquad \tilde{I}_R = \frac{V_R}{R} \angle \phi$$

the **impedance of the resistor** can be defined as:

$$Z_R = \frac{\tilde{V}_R}{\tilde{I}_R} = \frac{V_R \angle \phi}{\left(\frac{V_R}{R}\right) \angle \phi} = R \angle 0° = R + j0$$

Thus, the impedance of a resistor is equal to its resistance, which is a purely "real" value:

$$Z_R = R$$



$$v_R(t) = i_R(t) \cdot R$$

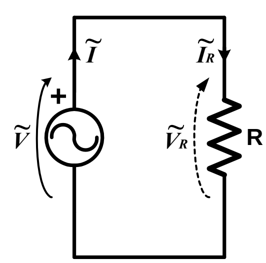
Impedance of a Resistor

Thus, not only does Ohm's Law hold true for resistors that are supplied with both DC voltages and time-varying (AC) voltages:

$$V_R = I_R \cdot R \qquad v_R(t) = i_R(t) \cdot R$$

Ohm's Law also holds true for resistors whose voltages and currents are expressed as phasors:

$$\tilde{V}_R = \tilde{I}_R \cdot Z_R = \tilde{I}_R \cdot R$$



$$v_R(t) = i_R(t) \cdot R$$

$$\tilde{V}_R = \tilde{I}_R \cdot R$$

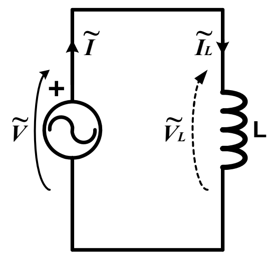
Impedance of an Inductor

Given the voltage across an inductor:

$$v_L(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

the current flowing through the inductor will be:

$$i_L(t) = \sqrt{2} \cdot \frac{V}{\omega \cdot L} \cdot \sin(\omega \cdot t + \phi° - 90°)$$



$$v_L(t) = L \cdot \frac{di_L(t)}{dt}$$

When expressed as phasors, the inductor's voltage and current can be rewritten as:

$$\tilde{V}_L = V_L \angle \phi \qquad \tilde{I}_L = \frac{V_L}{\omega \cdot L} \angle \phi - 90°$$

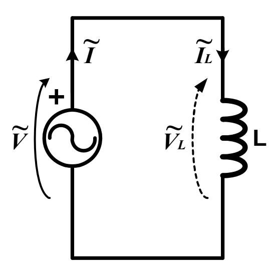
Impedance of an Inductor

Based on the inductor's phasor voltage and current:

$$\tilde{V}_L = V_L \angle \phi \qquad \tilde{I}_L = \frac{V_L}{\omega \cdot L} \angle \phi - 90°$$

the **impedance of the inductor** can be defined as:

$$Z_L = \frac{\tilde{V}_L}{\tilde{I}_L} = \frac{V_L \angle \phi}{\left(\frac{V_L}{\omega \cdot L}\right) \angle \phi - 90°} = (\omega \cdot L) \angle +90°$$



$$v_L(t) = L \cdot \frac{di_L(t)}{dt}$$

which can be expressed in rectangular form as:

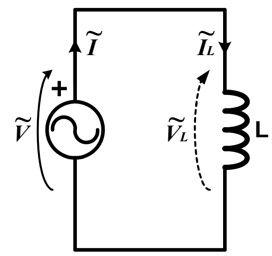
$$Z_L = (\omega \cdot L) \angle +90° = 0 + j\omega \cdot L = j\omega \cdot L$$

Impedance of an Inductor

Thus, based on its phasor voltage and current:

$$\tilde{V}_L = V_L \angle \phi \qquad \tilde{I}_L = \frac{V_L}{\omega \cdot L} \angle \phi - 90^\circ$$

the **impedance of the inductor** can be defined as:



$$Z_L = (\omega \cdot L) \angle +90^\circ = j\omega \cdot L$$

which is a <u>positive imaginary number</u>.

$$v_L(t) = L \cdot \frac{di_L(t)}{dt}$$

And, when expressed as impedances, Ohm's Law also holds true for inductors whose voltages and currents are expressed as phasors.

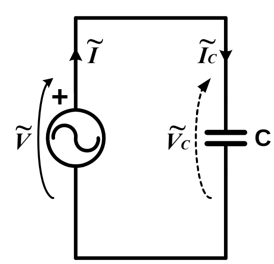
$$\tilde{V}_L = \tilde{I}_L \cdot Z_L$$

Impedance of an Capacitor

Given the voltage across a capacitor:

$$v_C(t) = \sqrt{2} \cdot V \cdot \sin(\omega \cdot t + \phi)$$

the current flowing through the capacitor will be:



$$i_C(t) = \sqrt{2} \cdot V \cdot \omega \cdot C \cdot \sin(\omega \cdot t + \phi^\circ + 90^\circ)$$

When expressed as phasors, the capacitor's voltage and current can be rewritten as:

$$i_C(t) = C \cdot \frac{dv_C(t)}{dt}$$

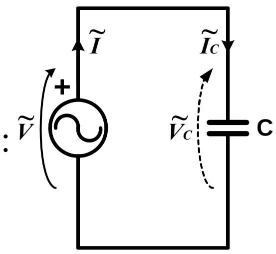
$$\tilde{V}_C = V_C \angle \phi \qquad \tilde{I}_C = V_C \cdot \omega \cdot C \angle \phi + 90^\circ$$

Impedance of an Capacitor

Based on the capacitor's phasor voltage and current:

$$\tilde{V}_C = V_C \angle \phi \qquad \tilde{I}_C = V_C \cdot \omega \cdot C \angle \phi + 90°$$

the **impedance of the capacitor** can be defined as:



$$Z_C = \frac{\tilde{V}_C}{\tilde{I}_C} = \frac{V \angle \phi}{V \cdot \omega \cdot C \angle \phi + 90°} = \frac{1}{\omega \cdot C} \angle -90°$$

$$i_C(t) = C \cdot \frac{dv_C(t)}{dt}$$

which can be expressed in rectangular form as:

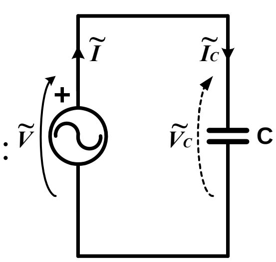
$$Z_C = \frac{1}{\omega \cdot C} \angle -90° = 0 - j\frac{1}{\omega \cdot C} = -j\frac{1}{\omega \cdot C}$$

Impedance of an Capacitor

Thus, based on its phasor voltage and current:

$$\tilde{V}_C = V_C \angle \phi \qquad \tilde{I}_C = V_C \cdot \omega \cdot C \angle \phi + 90°$$

the **impedance of the capacitor** can be defined as:



$$Z_C = \frac{1}{\omega \cdot C} \angle -90° = -j\frac{1}{\omega \cdot C}$$

which is a negative imaginary number.

$$i_C(t) = C \cdot \frac{dv_C(t)}{dt}$$

And, when expressed as impedances, Ohm's Law also holds true for capacitors whose voltages and currents are expressed as phasors.

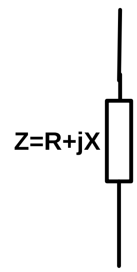
$$\tilde{V}_C = \tilde{I}_C \cdot Z_C$$

Complex Impedances

A complex impedance ***Z*** is an impedance can have both resistive and reactive (inductive or capacitive) components, and may be expressed in the form:

$$Z = R + jX$$

where: R is the resistive component of the load, and
X is the reactive component of the load.



Note:
- the impedance of a resistor is: $Z_R = R$
- the impedance of an inductor is: $Z_L = j(\omega \cdot L)$
- the impedance of a capacitor is: $Z_C = -j\left(\frac{1}{\omega \cdot C}\right)$

Reactance

Reactance defines the manner in which capacitive and inductive loads react to a steady-state sinusoidal voltage.

The reactance of a load is equal to the imaginary value of the load's impedance value.

Therefore:
- the reactance of a resistor is: $X_R = 0\ \Omega$
- the reactance of an inductor is: $X_L = \omega \cdot L\ \Omega$
- the reactance of a capacitor is: $X_C = \frac{-1}{\omega \cdot C}\ \Omega$

Phasor Analysis of AC Circuits

If all of the voltages and currents within a steady-state AC circuit are expressed as phasors:

$$\tilde{V} = V\angle\phi \qquad \tilde{I} = I\angle\delta$$

and all of the "loads" are defined by their impedance values:

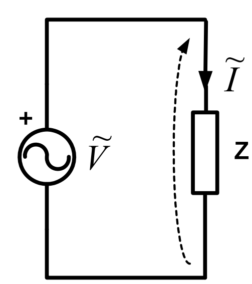
$$Z = R + jX$$

then the circuit's operation may be solved by a set of algebraic equations based on Ohm's Law:

$$\tilde{V} = \tilde{I}\cdot Z$$

Phasor Analysis of AC Circuits

If a voltage source having the phasor value $\tilde{V}$ is applied across the complex impedance Z, then the phasor value of the current $\tilde{I}$ may be solved by applying Ohm's Law:



$$\tilde{I} = \frac{\tilde{V}}{Z} = \frac{\tilde{V}}{R + jX} = I\angle\delta$$

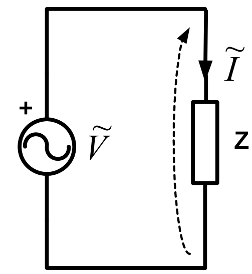
Phasor Analysis of AC Circuits

Similarly, given the voltage and current supplied to an impedance:

$$\tilde{V} = V\angle\phi \qquad \tilde{I} = I\angle\delta$$

the impedance may be defined in terms of voltage and current as:

$$Z = \frac{\tilde{V}}{\tilde{I}} = \frac{V\angle\phi}{I\angle\delta} = \frac{V}{I}\angle(\phi-\delta) = \frac{V}{I}\angle\theta = |Z|\angle\theta$$



Phasor Analysis of AC Circuits

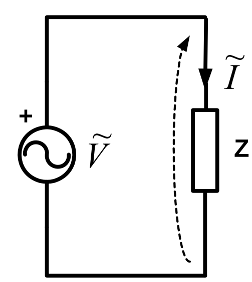
Thus, given:

$$\tilde{V} = V\angle\phi \qquad \tilde{I} = I\angle\delta$$

the impedance magnitude is defined by Ohm's Law and the impedance angle is the difference between the voltage and current angles.

$$Z = |Z|\angle\theta \qquad ==> \qquad |Z| = \frac{V}{I} \qquad \theta = \phi - \delta$$

Note that the impedance angle θ is the same as the previously defined "power angle" from the solution for AC power.



Phasor Analysis of AC Circuits

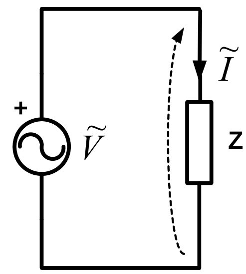
The following formulas may be used to convert an impedance between rectangular form and polar form:

$$Z = |Z|\angle\theta \quad ==> \quad Z = R + jX$$

$$\mathrm{R} = |Z| \cdot \cos(\theta) \qquad X = |Z| \cdot \sin(\theta)$$

$$Z = R + jX \quad ==> \quad Z = |Z|\angle\theta$$

$$|Z| = \sqrt{R^2 + X^2} \qquad \theta = \tan^{-1}\frac{X}{R}$$



Phasor Analysis of AC Circuits

Given the voltage and current:

$$\tilde{V} = V\angle\phi \qquad \tilde{I} = I\angle\delta$$

supplied to a complex impedance Z, the resultant angle θ may fall anywhere in the range:

$$-90° \leq \theta \leq +90°$$