ECET 2111

Circuits II

Mutually-Linked Coils

and

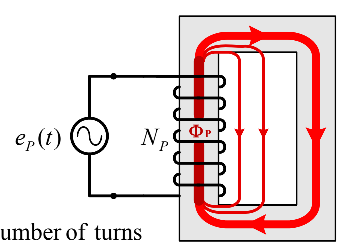
Transformers

AC-Supplied Coil

If a coil is supplied by an AC-source, $e_P(t)$, then a time-varying magnetic flux, Φ_P, will be created, as defined by:

$$e_P(t) = N_P \cdot \frac{d\Phi_P(t)}{dt} \quad \text{(Faraday's Law)}$$

the field lines of which will pass through the center of the coil and then back around the outside in order to form closed-loops.



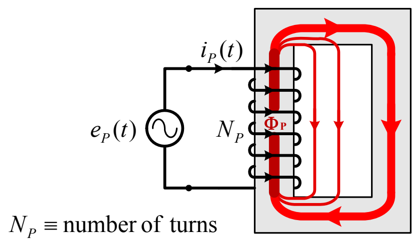
$N_P \equiv$ number of turns

AC-Supplied Coil

Additionally, the following relationship can be defined between the source voltage, $e_P(t)$, and the coil current, $i_P(t)$:

$$e_P(t) = L_P \cdot \frac{di_P(t)}{dt}$$

where: L_P is the self-inductance of the coil.

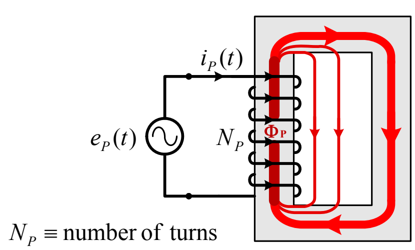


Self-Inductance

Note that a self-inductance, L_P, can be defined as:

$$L_P = N_P \cdot \frac{d\Phi_P(t)}{di_P(t)}$$

such that L_P is proportional to the rate of change in the flux created by the coil over the rate of change in the current flowing through the coil.

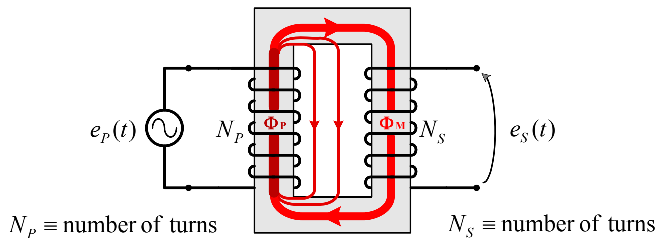


Mutually-Linked Coils

If a second coil is placed such that some of the flux passes through the coil, then a voltage will be induced across the second coil, also defined by:

$$e_S(t) = N_S \cdot \frac{d\Phi_M(t)}{dt} \quad \text{(Faraday's Law)}$$

where: Φ_M is the flux that passes through the second coil.

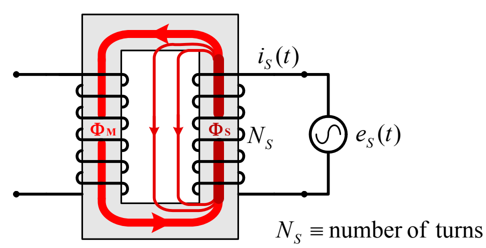


Self-Inductance

Note that, if an AC source is connected to the second coil, a self-inductance, L_S, can also be defined for that coil:

$$L_S = N_S \cdot \frac{d\Phi_S(t)}{di_S(t)} \quad \textbf{(Henries)}$$

such that L_S is proportional to the rate of change in the flux created by that coil over the rate of change that coil's current.

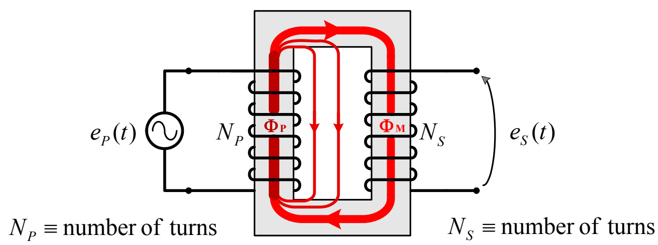


Coupling Factor

A coupling-factor, k, can be defined as:

$$k = \frac{\Phi_M(t)}{\Phi_P(t)}$$

such that k is the ratio of the flux created by the first coil that passes through the second coil over the total flux created by the first coil.

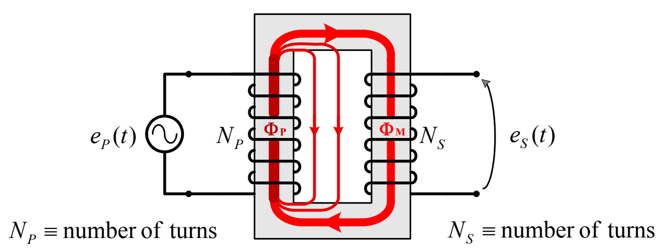


Mutual Inductance

Furthermore, a mutual inductance, M, can be defined as:

$$M = N_S \cdot \frac{d\Phi_M(t)}{di_P(t)} \quad \textbf{(Henries)}$$

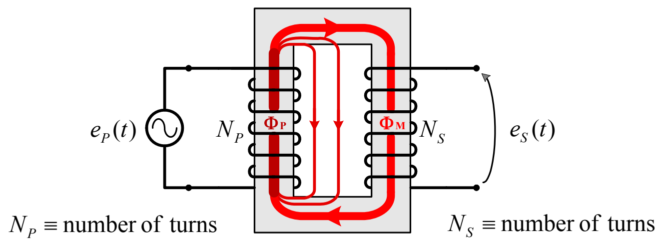
such that M is proportional to the rate of change in the flux that passes through the second coil over the rate of change in the current flowing through the first coil.



Mutual Inductance

It also turns out that the mutual inductance, M, can be expressed in terms of the self-inductances of the coils as:

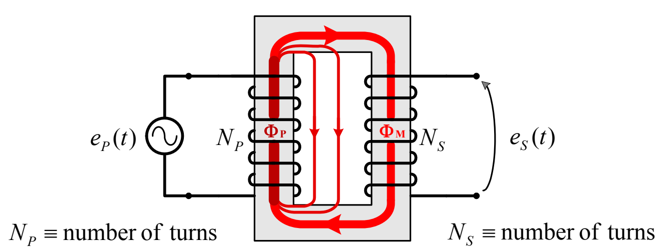
$$M = k \cdot \sqrt{L_P \cdot L_S} \qquad \textbf{(Henries)}$$



Mutually-Linked Coils

Note that the voltage, $e_S(t)$, induced across the second coil by the mutually-linked flux created by the first coil can be expressed in terms of the mutual inductance as follows:

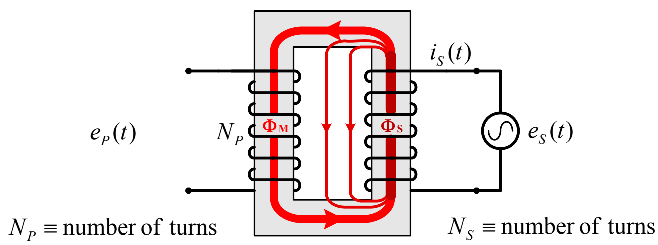
$$e_S(t) = M \cdot \frac{di_P(t)}{dt} \qquad \textbf{(volts)}$$



Mutually-Linked Coils

It also turns out that, if a current is flowing in the second coil, a flux will be created by the second coil that passes through the first coil and induces a voltage, $e_P(t)$, across that coil, as defined by:

$$e_P(t) = M \cdot \frac{di_S(t)}{dt} \quad \text{(volts)}$$

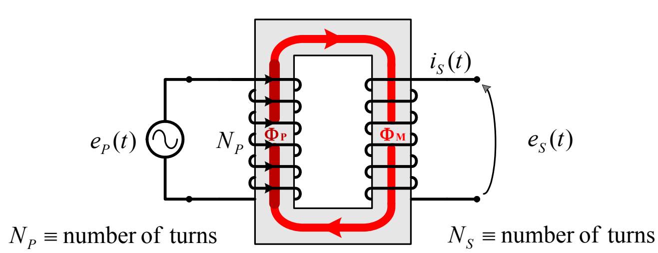


Iron-Core Transformers

An "ideal" iron-core transformer consists of two coils that are mutually-linked by an iron core that provides an "ideal" closed-loop path for the flux created by the first coil.

If all of the flux stays within the iron core, then all of the flux created by the first coil will pass through the second coil:

$$\Phi_M(t) = \Phi_P(t) = \Phi_S(t)$$

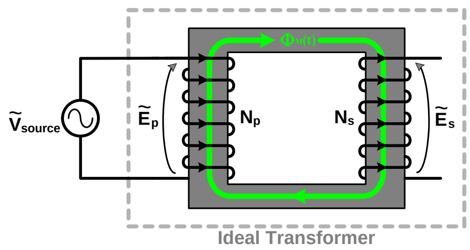


Mutually Linked Coils

If the magnetic core is assumed to be ideal, then the total flux created by the sourced coil will pass through the second coil.

Since a time-varying flux passes through the second coil, a voltage will be induced across that coil, also defined by:

$$\tilde{E}_s = N_s \cdot \frac{d\Phi_M(t)}{dt}$$

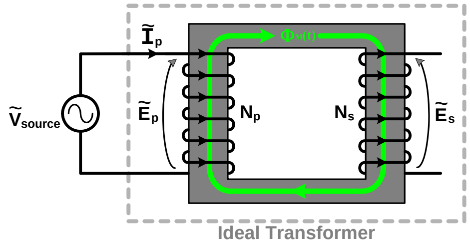


Mutually Linked Coils

If the total flux passes through both coils, then the rate of change, $\frac{d\Phi(t)}{dt}$, of the flux through the coils must be the same.

The following relationship may be derived by solving for $\frac{d\Phi(t)}{dt}$ in both coils and equating the results:

$$\frac{\tilde{E}_s}{N_s} = \frac{d\Phi_M(t)}{dt} = \frac{\tilde{E}_p}{N_p}$$

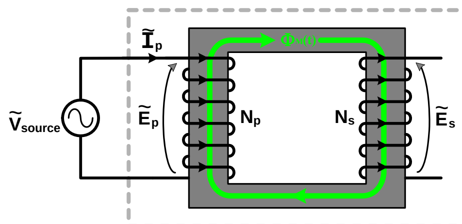


Voltage Relationship

The relationship between the two coil voltages is typically expressed as a ratio of the voltages, which equals to the ratio of their respective number of turns.

(I.e. – the “turns ratio” of the transformer).

$$\frac{\tilde{E}_p}{N_p} = \frac{\tilde{E}_s}{N_s} \quad \Rightarrow \quad \boxed{\frac{\tilde{E}_p}{\tilde{E}_s} = \frac{N_p}{N_s}}$$



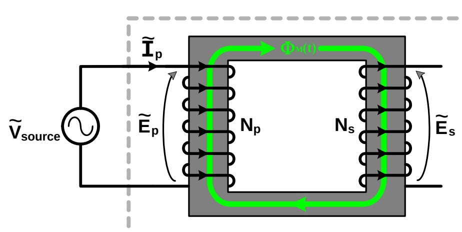
Ideal Transformer

Turns-Ratio

The ratio relationship, referred to as the <u>turns ratio</u> (*a*):

$$a = \frac{\tilde{E}_p}{\tilde{E}_s} = \frac{N_p}{N_s}$$

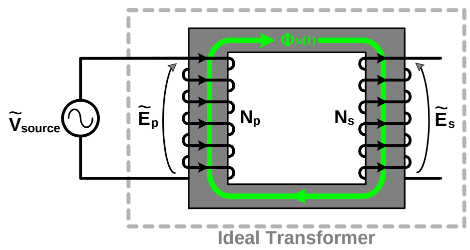
defines the basic operation of an ideal transformer in terms of the primary and secondary voltages.



Ideal Transformer

Polarity Relationship

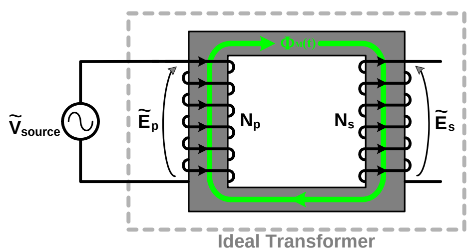
Note – the polarity of the voltage induced across the second coil is based upon both the direction of the flux within the core and the direction that the coil is wrapped around the core.



Lenz's Law

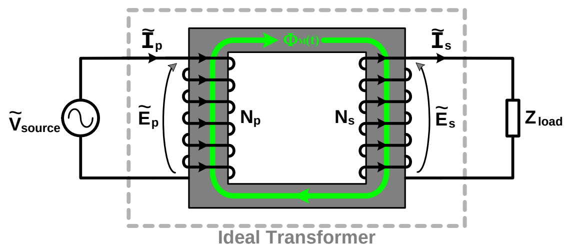
The correct polarity relationship can be determined by applying Lenz's Law, which states:

"Any induced effect will always oppose its source."



Determining the Polarity Relationship

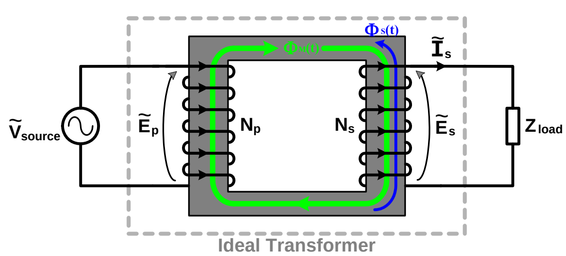
If a load is connected to the second winding, then a current will flow out of the secondary winding and through the load due to the induced voltage.



Determining the Polarity Relationship

If a load is connected to the second winding, then a current will flow out of the secondary winding and through the load due to the induced voltage.

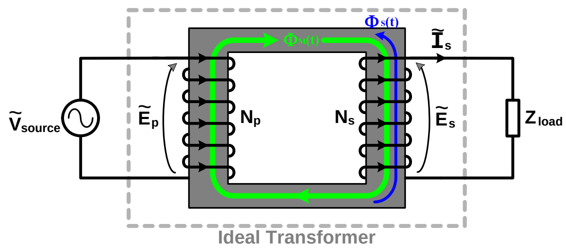
Based on Lenz's Law, the polarity of the voltage must be such that the resultant coil-current will create a counter-flux in the core that opposes the original flux.



Secondary Current Effects

But, the existence of a counter-flux produced by the current that is flowing in the second coil would tend to decrease the overall flux within the magnetic core, in-turn decreasing the total flux passing through the primary coil.

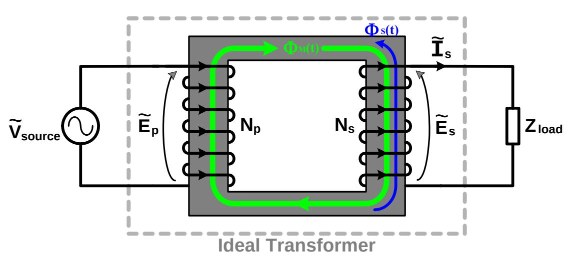
$$\Phi_{Net} = \Phi_M - \Phi_S$$



Secondary Current Effects

Assuming that the source is ideal, this presents a problem because Faraday's Law does not allow for a change in the flux passing through the primary coil unless the supply voltage changes accordingly.

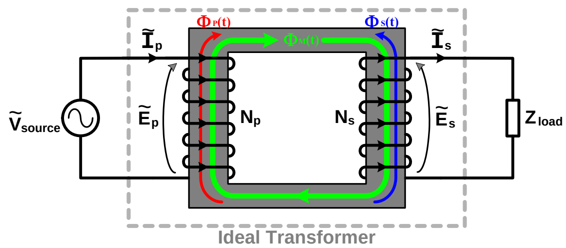
$$\tilde{E}_P = N_P \cdot \frac{d\Phi_{Net}(t)}{dt}$$



Secondary Current Effects

Thus, the existence of the secondary current's counter-flux requires that a additional (primary) current be drawn into the primary winding.

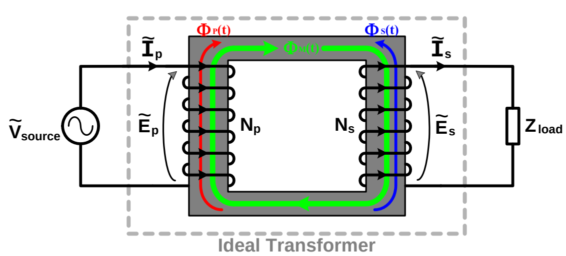
The primary current will, in-turn, create an additional flux component, Φ_p, within the core that is equal in magnitude but opposite in direction compared to the secondary flux Φ_s.



Primary Current

Since the primary and secondary fluxes are equal in magnitude but opposite in direction, they will cancel, leaving the net flux in the core the same as defined by Faraday's Law applied to the primary winding:

$$\Phi_{Net} = \Phi_M - \Phi_S + \Phi_P = \Phi_M$$



Primary/Secondary Current Ratio

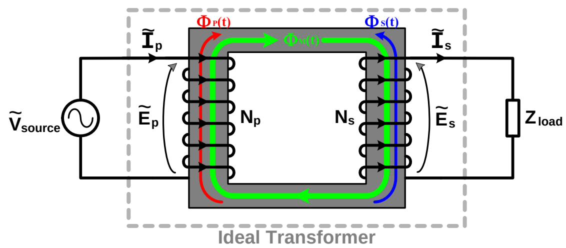
Based on the MMF relationship applied to both coils:

$$N \cdot i(t) = \Phi(t) \cdot \Re$$

the ratio of the primary and secondary currents must be:

$$\frac{\tilde{I}_p}{\tilde{I}_s} = \frac{1}{a}$$

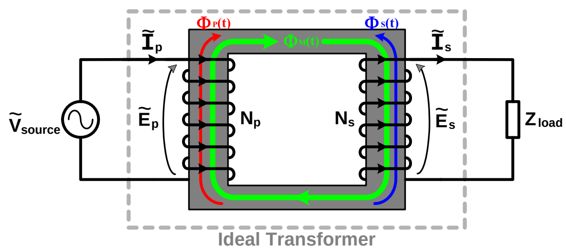
in order for their fluxes to cancel.



Overall Operation of Ideal Transformer

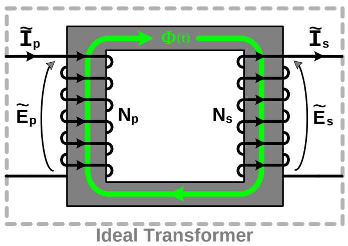
Thus, the overall operation of the ideal transformer that supplies a single load can be defined by the following set of equations:

turns ratio $a = \dfrac{N_p}{N_s}$ $\quad a = \dfrac{\tilde{E}_p}{\tilde{E}_s}$ $\quad \dfrac{\tilde{I}_p}{\tilde{I}_s} = \dfrac{1}{a}$ $\quad \tilde{I}_s = \dfrac{\tilde{E}_s}{Z_{load}}$

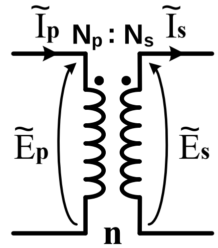


Ideal Transformer Equivalent Circuit

The following equivalent circuit will be used to represent an ideal transformer:



$\leftrightarrow$



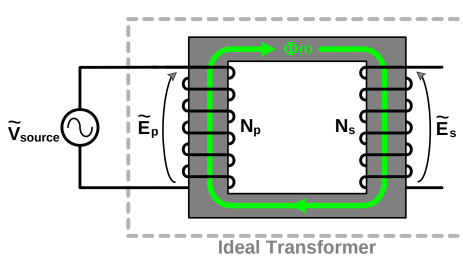
$$a = \frac{N_p}{N_s} \qquad \frac{\tilde{E}_p}{\tilde{E}_s} = a \qquad \frac{\tilde{I}_p}{\tilde{I}_s} = \frac{1}{a} \qquad a = \text{turns ratio}$$

Ideal Transformer Definitions

Primary Winding ≡ the winding that creates the mutually-linked flux (I.e. – the sourced winding).

Secondary Winding ≡ the winding across which a voltage is induced (I.e. – the load winding).

Note – the primary & secondary winding designations can also be defined in terms of the power flow direction (I.e. – the source & load connections)

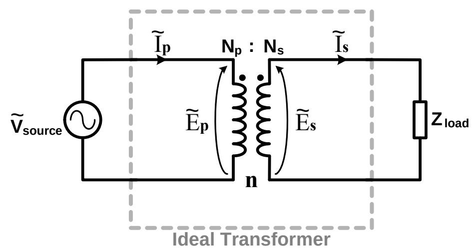


Ideal Transformer Definitions

High-Voltage Winding ≡ the winding with the larger voltage magnitude.
(I.e. – the coil with the larger number of turns)

Low-Voltage Winding ≡ the winding with the smaller voltage magnitude.
(I.e. – the coil with the smaller number of turns)

Note – the high-voltage winding will have the larger number of turns while the low-voltage winding will have the smaller number of turns.



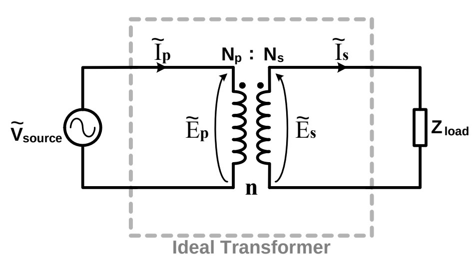
Ideal Transformer Definitions

Step-Up Transformer ≡ a transformer whose <u>voltage</u> increases from primary to secondary winding.

Step-Down Transformer ≡ a transformer whose <u>voltage</u> decreases from primary to secondary winding.

Notes: A step-up transformer's turns ratio will be less than one (*a*<1).

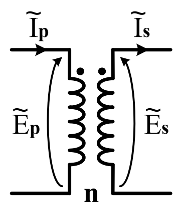
A step-down transformer's turns ratio will be greater than one (*a*>1).



Polarity Relationship

The polarity relationship between the primary and secondary voltages depends on the direction that the coils are wrapped around the magnetic core.

The "Dot Convention" is often used to provide the polarity relationship for a specific transformer.

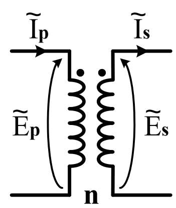


Equivalent Circuit "Dot Convention"

"Dots" are often included with the equivalent circuit to define the polarity relationship between the transformer windings.

1) An applied primary voltage whose voltage-rise points toward the primary winding's dot will induce a secondary voltage whose voltage-rise points toward the secondary winding's dot.

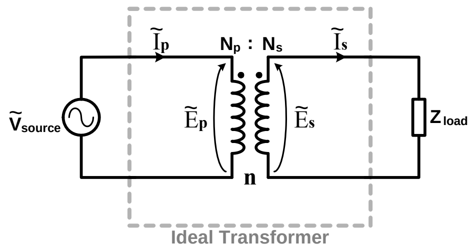
2) A current will be drawn <u>into</u> the dot side of the primary winding when a current flows <u>out of</u> the dot side of the secondary winding.



Turns-Ratio Consideration

It is important to note that the turns ratio, *a*, will change depending on which of the two windings are utilized as the primary winding.

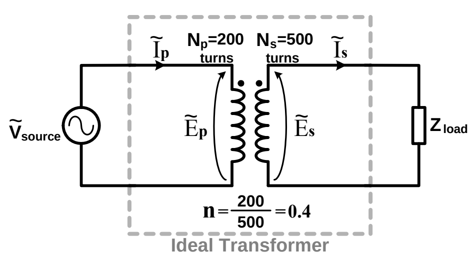
$$a = \frac{N_p}{N_s} \qquad \frac{\tilde{E}_p}{\tilde{E}_s} = a \qquad \frac{\tilde{I}_p}{\tilde{I}_s} = \frac{1}{a} \qquad a = \text{turns ratio}$$



Turns-Ratio Consideration

For Example – Given a transformer with a 200-turn winding and a 500-turn winding:

The transformer will have a turns ratio *a*=0.4 if the 200-turn winding is used as the primary, or

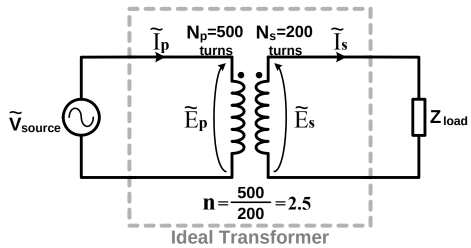


Turns-Ratio Consideration

For Example – Given a transformer with a 200-turn winding and a 500-turn winding:

The transformer will have a turns ratio a=0.4 if the 200-turn winding is used as the primary, or

The transformer will have a turns ratio a=2.5 if the 500-turn winding is used as the primary.



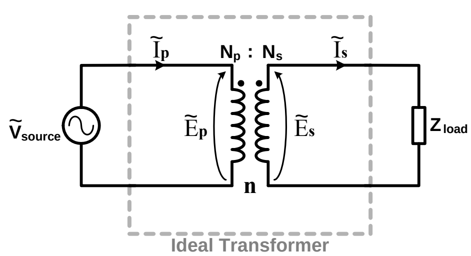
Turns-Ratio Consideration

Based on the voltage ratio:

$$\frac{\tilde{E}_p}{\tilde{E}_s} = \frac{N_p}{N_s} = a$$

it can be seen that:

- **$E_p > E_s$ when $a > 1$ (step-down operation), and**
- **$E_s > E_p$ when $a < 1$ (step-up operation).**



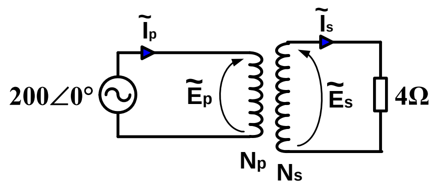
Ideal Transformer Example Problem

Given a transformer that contains windings having 50 and 500 turns:

If a 200∠0° volt source is connected across the transformer's 50-turn winding and a 4Ω load is connected across the 500-turn winding,

Determine:

- **The load voltage,**
- **The load current,**
- **The real power consumed by the load,**
- **The source current, and**
- **The real power produced by the source.**

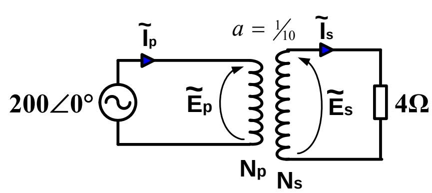


Ideal Transformer Example Problem

If a 200∠0° volt source is connected across the transformer's 50-turn winding and a 4Ω load is connected across the 500-turn winding:

The turns-ratio for the transformer, as connected, is:

$$a = \frac{N_p}{N_s} = \frac{50}{500} = \frac{1}{10}$$



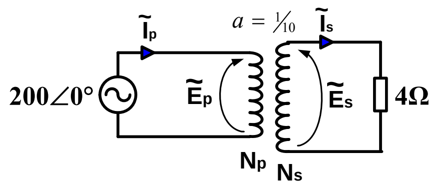
Ideal Transformer Example Problem

If a 200∠0° volt source is connected across the transformer's 50-turn winding and a 4Ω load is connected across the 500-turn winding…

Since the source is directly connected to the primary winding, the primary winding voltage $\tilde{E}_p$ is equal to the source voltage, thus:

$$\tilde{E}_p = 200\angle 0°$$

And, since the load is connected directly to the secondary winding, the load voltage and current are equal to $\tilde{E}_s$ and $\tilde{I}_s$ respectively.

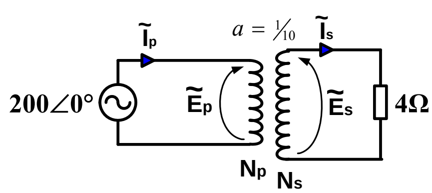


Ideal Transformer Example Problem

If a 200∠0° volt source is connected across the transformer's 50-turn winding and a 4Ω load is connected across the 500-turn winding…

The secondary (load) voltage $\tilde{E}_s$ can be determined from the equation:

$$\tilde{V}_{load} = \tilde{E}_s = \frac{\tilde{E}_p}{a} = \frac{200\angle 0°}{\frac{1}{10}} = 2{,}000\angle 0° \text{ volts}$$

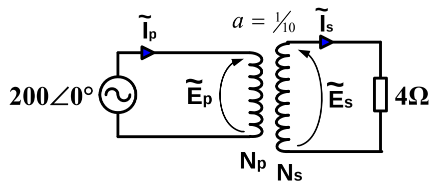


Ideal Transformer Example Problem

If a 200∠0° volt source is connected across the transformer's 50-turn winding and a 4Ω load is connected across the 500-turn winding…

The resultant secondary (load) current $\tilde{I}_s$ will be:

$$\tilde{I}_{load} = \tilde{I}_s = \frac{\tilde{V}_{load}}{Z_{load}} = \frac{2{,}000\angle 0^\circ}{4} = 500\angle 0^\circ \text{ amps}$$



Ideal Transformer Example Problem

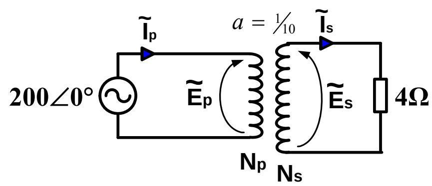
If a 200∠0° volt source is connected across the transformer's 50-turn winding and a 4Ω load is connected across the 500-turn winding…

The complex power, S_{load}, consumed by the load will be:

$$S_{load} = \tilde{V}_{load} \cdot \tilde{I}^*_{load} = (2{,}000\angle 0^\circ)\cdot(500\angle 0^\circ) = \boxed{1{,}000{,}000} + j0$$

from which the load power can be determined:

$$P_{load} = 1{,}000{,}000 \text{ watts}$$

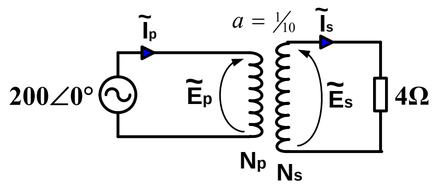


Ideal Transformer Example Problem

If a 200∠0° volt source is connected across the transformer's 50-turn winding and a 4Ω load is connected across the 500-turn winding…

The primary (source) current $\tilde{I}_p$ can be determined from the equation:

$$\tilde{I}_{source} = \tilde{I}_p = \frac{\tilde{I}_s}{a} = \frac{500\angle 0°}{\frac{1}{10}} = 5{,}000\angle 0° \text{ amps}$$



Ideal Transformer Example Problem

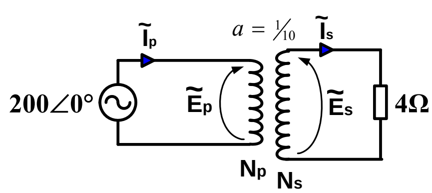
If a 200∠0° volt source is connected across the transformer's 50-turn winding and a 4Ω load is connected across the 500-turn winding…

Finally, the complex power, S_{source}, produced by the source will be:

$$S_{source} = \tilde{V}_{source} \cdot \tilde{I}^*_{source} = (200\angle 0°)\cdot(5{,}000\angle 0°) = \boxed{1{,}000{,}000} + j0$$

from which the source power can be determined:

$$P_{source} = 1{,}000{,}000 \text{ watts}$$

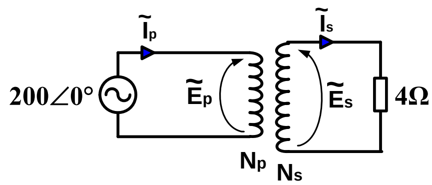


Ideal Transformer Example Problem

Given a transformer that contains windings having 50 and 500 turns:

If a 200∠0° volt source is connected across the transformer's 50-turn winding and a 4Ω load is connected across the 500-turn winding:

- **Load Voltage:** $\tilde{V}_{load} = 2{,}000\angle 0°$ volts
- **Load Current:** $\tilde{I}_{load} = 500\angle 0°$ amps
- **Load Power:** $P_{load} = 1{,}000{,}000$ watts
- **Source Current:** $\tilde{I}_{source} = 5{,}000\angle 0°$ amps
- **Source Power:** $P_{source} = 1{,}000{,}000$ watts

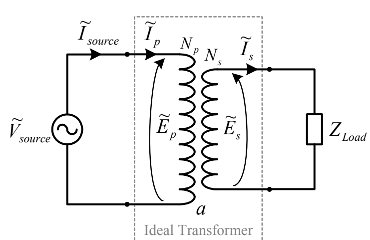


Input Impedance

Given an ideal transformer with a source connected across the primary winding and a load connected across the secondary winding…

Determine the overall impedance "seen" by the source.

(I.e. – the input impedance of the ideal transformer)



$$a = \frac{N_p}{N_s}$$

$$\frac{\tilde{E}_p}{\tilde{E}_s} = a$$

$$\frac{\tilde{I}_p}{\tilde{I}_s} = \frac{1}{a}$$

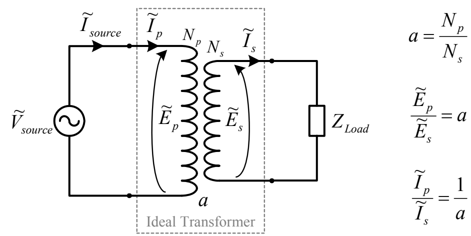
Input Impedance

The input impedance of the transformer may be defined as:

$$Z_{in} = \frac{\tilde{E}_p}{\tilde{I}_p}$$

If we substitute the following relations into the equation:

$$\tilde{E}_p = a \cdot \tilde{E}_s \qquad \tilde{I}_p = \tfrac{1}{a} \cdot \tilde{I}_s$$



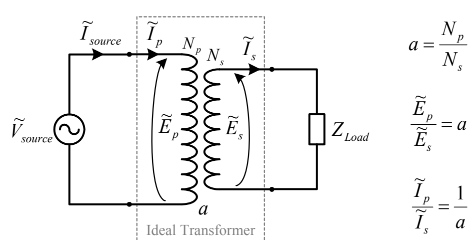
Input Impedance

Then the input impedance may be re-defined as:

$$Z_{in} = \frac{a \cdot \tilde{E}_s}{\frac{1}{a} \cdot \tilde{I}_s} = a^2 \cdot \frac{\tilde{E}_s}{\tilde{I}_s}$$

Since $\tilde{E}_s$ and $\tilde{I}_s$ equal the load voltage and current respectively:

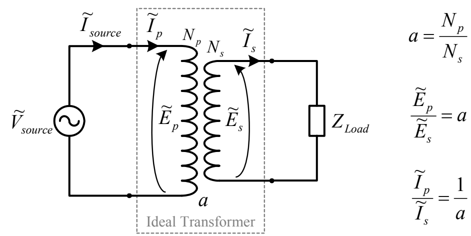
$$\frac{\tilde{E}_s}{\tilde{I}_s} = \frac{\tilde{V}_{Load}}{\tilde{I}_{Load}} = Z_{Load}$$



Input Impedance

If expressed in terms of the load impedance, the input impedance of the ideal transformer is:

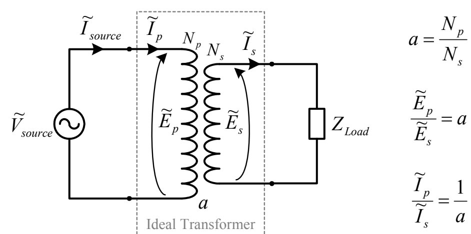
$$Z_{in} = a^2 \cdot \frac{\tilde{E}_s}{\tilde{I}_s} = a^2 \cdot Z_{Load}$$



Input Impedance

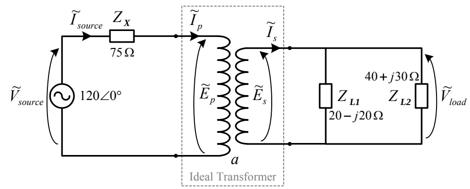
Thus, the input impedance of an ideal transformer is equal to its turns-ratio squared times the impedance of its connected load:

$$Z_{in} = a^2 \cdot Z_{Load} = Z'_{Load}$$



Ideal Transformer Example Problem

Given the following circuit that contains a 120V–48V ideal transformer:

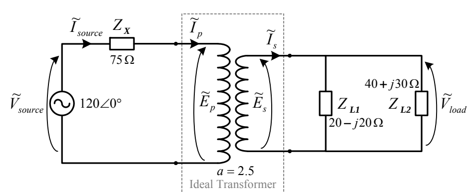


Assuming that the transformer is connected such that it operates as a "step-down" transformer,

Determine:
- **The source current, and**
- **The load voltage.**

Ideal Transformer Example Problem

Given the following circuit that contains a 120V–48V ideal transformer:

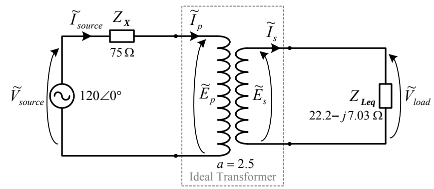


Based on "step-down" operation, the transformer's turns-ratio is:

$$a = \frac{V_{Rated(\text{Pri})}}{V_{Rated(\text{Sec})}} = \frac{120\text{V}}{48\text{V}} = 2.5$$

Ideal Transformer Example Problem

Given the following circuit that contains a 120V–48V ideal transformer:



Based on "step-down" operation, the transformer's turns-ratio is:

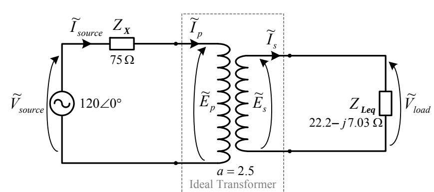
$$a = \frac{V_{Rated\,(\mathrm{Pri})}}{V_{Rated\,(\mathrm{Sec})}} = \frac{120\mathrm{V}}{48\mathrm{V}} = 2.5$$

And, since Z_{L1} is connected in parallel with Z_{L2}:

$$Z_{Leq} = \left(\frac{1}{Z_{L1}} + \frac{1}{Z_{L2}}\right)^{-1} = \left(\frac{1}{20 - j20} + \frac{1}{40 + j30}\right)^{-1} = (22.2 - j7.03)\ \Omega$$

Ideal Transformer Example Problem

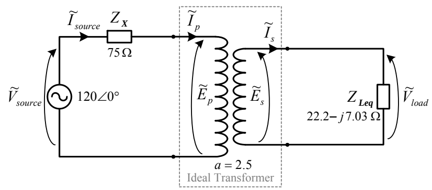
Given the following circuit that contains a 120V–48V ideal transformer:

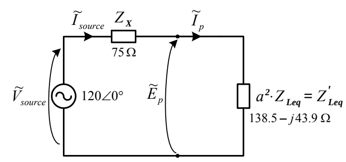


Since the source is not directly connected to the primary winding, it would be easier to first simplify the circuit by replacing the ideal transformer and load combination with the equivalent input impedance seen looking into the transformer's primary terminals:

Ideal Transformer Example Problem

Given the following circuit that contains a 120V−48V ideal transformer:



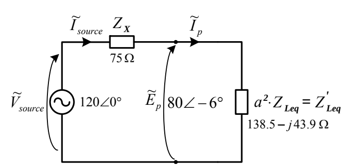


Since the source is not directly connected to the primary winding, it would be easier to first simplify the circuit by replacing the ideal transformer and load combination with the equivalent input impedance seen looking into the transformer's primary terminals:

$$Z_{in} = Z'_{Leq} = a^2 \cdot Z_{Leq} = 2.5^2 \cdot (22.2 - j7.03) = (138.5 - j43.9)\ \Omega$$

Ideal Transformer Example Problem

Given the following circuit that contains a 120V−48V ideal transformer:



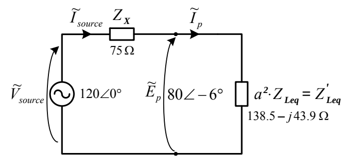
Now that the circuit has been reduced-down to two series impedances:

$$\boxed{\tilde{I}_{source}} = \frac{\tilde{V}_{source}}{(Z_x + Z'_{Leq})} = \frac{120\angle 0°}{75 + (138.5 - j43.9)} = \boxed{0.5505\angle 11.6°\ amps} = \tilde{I}_p$$

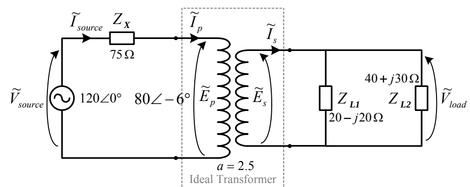
$$\tilde{E}_p = \tilde{I}_p \cdot Z'_{Leq} = (0.5505\angle 11.6°) \cdot (138.5 - j43.9) = 80\angle -6° \text{ volts}$$

Ideal Transformer Example Problem

Given the following circuit that contains a 120V−48V ideal transformer:

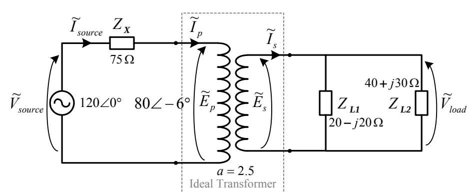


But E_p in the reduced circuit must equal to E_p in the original circuit.



Ideal Transformer Example Problem

Given the following circuit that contains a 120V−48V ideal transformer:



But E_p in the reduced circuit must equal to E_p in the original circuit.

Thus, given E_p, in E_s (which is also equal to V_{load}) can be determined by:

$$\boxed{\tilde{V}_{load}} = \tilde{E}_s = \frac{\tilde{E}_p}{a} = \frac{80\angle -6°}{2.5} = \boxed{32\angle -6° \text{ volts}}$$